A TEXTBOOK OF
ENGINEERING MATHEMATICS
(For M.D.U., G.J.U. and K.U.K., Haryana)
SEMESTER-III/IV

N.P. BALI
A TEXTBOOK OF
ENGINEERING
MATHEMATICS

For
B.E. II Year (III/IV Semester)
M.D.U., G.J.U. and K.U., Haryana
(Strictly According to New Syllabus)

By
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S.B. College, Gurgaon
Haryana

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SYLLABUS
(M.D.U., Rohtak)

MAT-201-F MATHEMATICS-III

L T P
3 1 1

Class Work : 50 Marks
Theory : 100 Marks
Total : 150 Marks
Duration of Exam : 3 Hours

Note: Examiner will set 9 questions in total, with two questions from each section and one question covering all sections which will be Q 1. This Q 1 is compulsory and of short answers type. Each question carries equal mark (20 marks). Students have to attempt 5 questions in total at least one question from each section.

SECTION-A

Fourier Series and Fourier Transforms: Euler’s formulae, conditions for a Fourier expansion, change of interval, Fourier expansion of odd and even functions, Fourier expansion of square wave, rectangular wave, saw-toothed wave, half and full rectified wave, half range sine and cosine series.

Fourier integrals, Fourier transforms, Shifting theorem (both on time and frequency axes), Fourier transforms of derivatives, Fourier transforms of integrals, Convolution theorem, Fourier transform of Dirac-delta function.

SECTION-B

Functions of Complex Variable: Definition, Exponential function, Trigonometric, Hyperbolic and Logarithmic functions, Limit and Continuity of a function, Differentiability and Analyticity.

Cauchy-Riemann equations, necessary and sufficient conditions for a function to be analytic, polar form of the Cauchy-Riemann equations. Harmonic functions, application to flow problems. Integration of complex functions, Cauchy-Integral theorem and formula.

SECTION-C

Power series, radius and circle of convergence, Taylor’s, Maclaurin’s and Laurent’s series. Zeros and singularities of complex functions, Residues. Evaluation of real integrals using residues (around unit and semi circle only).

Probability Distributions and Hypothesis Testing: Conditional probability, Bayes’s theorem and its applications, expected value of a random variable. Properties and application of Binomial, Poisson and Normal distributions.

SECTION-D

Testing of a hypothesis, tests of significance for large samples, Student’s t-distribution (applications only). Chi-square test of goodness of fit.

Linear Programming: Linear programming problems formulation, solving linear programming problems using (i) Graphical method (ii) Simplex method (iii) Dual simplex

K.U., Kurukshetra

Theory : 100 Marks
Sessional : 50 Marks
Total : 150 Marks
Duration of Exam : 3 Hours

UNIT-I


UNIT-II

Functions of a Complex Variable: Functions of a complex variable, Exponential function, Trigonometric, Hyperbolic and Logarithmic functions, limit and continuity of a function, Differentiability and analyticity.

Cauchy-Riemann Equations: Necessary and sufficient conditions for a function to be analytic, Polar Form of the Cauchy-Riemann equations, Harmonic functions, Application to flow problems, Conformal transformation, Standard transformations (Translation, Magnification and rotation, inversion and reflection, Bilinear)

UNIT-III


UNIT-IV

Linear Programming: Linear programming problems formulation, Solution of Linear Programming Problem using Graphical method. Simplex Method, Dual-Simplex Method.
Basic Concepts

1. Greek Letters Used

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2. Some Notations

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3. Useful Data

- $\sqrt{2} = 1.4142$
- $\frac{1}{\pi} = 0.3183$
- $1 \text{ rad.} = 57^\circ 17' 45''$
- $\log_{10} e = 0.4343$
- $\sqrt{3} = 1.732$
- $e = 2.7183$
- $1' = 0.0174 \text{ rad.}$
- $\log_{e} 2 = 0.6931$
- $\pi = 3.1416$
- $\frac{1}{e} = 0.3679$
- $\log_{10} 10 = 2.3026$
- $\log_{10} 3 = 1.0986$

4. Quadratic Equation

Roots of quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$

are $\frac{-b \pm \sqrt{D}}{2a}$, where $D = b^2 - 4ac$ is called discriminant.

Sum of roots = $-\frac{b}{a}$, Product of roots = $\frac{c}{a}$

If $D > 0$, roots are real and distinct.
If $D = 0$, roots are equal.
If $D < 0$, roots are imaginary.

5. Progressions

(i) For the A.P. (Arithmetic Progression) $a, a + d, a + 2d, \ldots$

$T_n = a + (n - 1)d$, $S_n = \frac{n}{2} [2a + (n - 1)d] = \frac{n}{2} (a + l)$.

(ii) For the G.P. (Geometric Progression) $a, ar, ar^2, \ldots$

$T_n = ar^{n-1}$, $S_n = \begin{cases} \frac{a(1 - r^n)}{1 - r} & \text{when } r \neq 1 \\ na & \text{when } r = 1 \end{cases}$

6. Permutations and Combinations

- $\binom{n}{r} = \frac{n!}{(n-r)!}$
- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

7. Binomial Theorem

(i) When $n$ is a positive integer

$(a + b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}b^n$

(ii) When $n$ is a negative integer or a fraction

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots $...

provided $|x| < 1$.

8. Logarithms

(i) Natural logarithm of a positive real number $x$ is denoted by $\log_{e} x$ or simply $\log x$ or $\ln x$. It is the inverse of $e^x$.

Common logarithm of a positive real number $x$ is denoted by $\log_{10} x$.

Relation: \( (i) \ log_{a} x = x \Rightarrow \ log_{b} x = \frac{\log_{a} x}{\log_{b} a} \)

(ii) $\log_1 1 = 0.0$ \(, \ log_{a} a = 1 \), \(, \ log_{a} 0 = -\infty \) \( (a > 1) \)

(iii) $\log(mn) = \log m + \log n$, $\log \left( \frac{m}{n} \right) = \log m - \log n$

\(, \ log(m^n) = n \log m \), $\log_{a} m \times \log_{a} n = 1$

9. Matrices and Determinants

(i) Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if they have same order and corresponding elements are equal, i.e., $a_{ij} = b_{ij}$ for all $i$ and $j$.

(ii) If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, then $A + B$ is defined and $A + B = [a_{ij} + b_{ij}]$, i.e., add corresponding elements.

(iii) If $A = [a_{ij}]$ is a matrix and $k$ is a scalar, then $kA$ is another matrix obtained by multiplying each element of $A$ by the scalar $k$. Thus, $kA = [ka_{ij}]$. 

\( (x) \)

$$S_n = \frac{a}{1 - r} \text{ provided } |r| < 1 \ i.e., \ -1 < r < 1$$

(iii) A sequence is said to be in H.P. (Harmonic Progression) if the reciprocals of its terms are in A.P.

For the H.P. $\frac{1}{a}, \frac{1}{a + d}, \frac{1}{a + 2d}, \ldots$, $T_n = \frac{1}{a + (n-1)d}$

(iv) For two numbers $a$ and $b$,

- $A.M. = \frac{a + b}{2}$
- $G.M. = \sqrt{ab}$
- $H.M. = \frac{2ab}{a + b}$

(v) For natural numbers $1, 2, 3, \ldots, n$

$$\sum n = \frac{n(n + 1)}{2}, \sum n^2 = \frac{n(n + 1)(2n + 1)}{6}, \sum n^3 = \left(\frac{n(n + 1)}{2}\right)^2$$
where co-ratio of $x$ is obtained by dropping co if present and adding co if absent. Thus, $\sin \rightarrow \cos, \tan \rightarrow \cot, \sec \rightarrow \cosec.$

The sign $+$ or $-$ is decided from the quadrant in which $n \cdot 90^\circ + x$ lies.

(iii) Signs of $r$-ratios in different quadrants (Fig. 2)

(vi) $\cot^2 x - \cos^2 x = 1, \sec^2 x - \cos^2 x = 1$

(vii) $\sin (x + y) = \sin x \cos y + \cos x \sin y$
$\cos (x + y) = \cos x \cos y - \sin x \sin y$

$\tan (x + y) = \frac{\tan x \tan y}{1 - \tan x \tan y}$

(viii) $3 \tan x - 3 \tan^3 x = \frac{\tan x}{1 - 3 \tan^2 x}$

(ix) $\sin x \cos y = \frac{1}{2} [\sin(x + y) + \sin(x - y)]$

$\cos x \sin y = \frac{1}{2} [\sin(x + y) - \sin(x - y)]$

$\cos x \cos y = \frac{1}{2} [\cos(x + y) + \cos(x - y)]$

$\sin x \sin y = \frac{1}{2} [\cos(x + y) - \cos(x - y)]$

$x + y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$

$x - y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2}$

$\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$

$\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$

(a) $\sin a + b \cos b = r \sin (a + b), a \cos x + b \sin x = r \cos (x - b), a = r \cos b, b = r \sin b$

so that $r = \sqrt{a^2 + b^2}, \theta = \tan^{-1} \left( \frac{b}{a} \right)$

(iii) In any $\Delta ABC$:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

(sine formula)

(cosine formula)

$$\frac{a}{b} \cos C + c \cos B$$

(projection formula)

Area of $\Delta ABC = \frac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)},$ where $s = \frac{1}{2} (a + b + c)$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$ where $R$ is the circum-radius of $\Delta ABC.$

$$R = \frac{abc}{4A}$$

where $r$ is the radius of inscribed circle of $\Delta ABC.$

11. De Moivre's Theorem: $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

Euler's Theorem: $\cos \theta + i \sin \theta = e^{i\theta}$

12. Hyperbolic Functions

$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}, \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$cosech x = \frac{1}{\sinh x}, \sech x = \frac{1}{\cosh x}, \coth x = \frac{1}{\tanh x}$

$\tanh x = \frac{\sinh x}{\cosh x}, \coth x = \frac{\cosh x}{\sinh x}, \cosh^2 x - \sinh^2 x = 1$

$\sinh x = i \sin x, \cos x = \cos x, \tan x = i \tan x$

$\sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right), \cosh^{-1} x = \log \left( x + \sqrt{x^2 - 1} \right)$

$tanh^{-1} x = \frac{1}{2} \log \left( \frac{1+x}{1-x} \right)$

13. Series

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \ldots \infty$

$\sin x = \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots \ldots \infty, \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \ldots \infty$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \ldots \infty, \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \ldots \infty$

$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \ldots \infty, \log (1 - x) = \left( x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots \ldots \infty \right)$

$\tan^{-1} x = \frac{x^3}{3} + \frac{x^5}{5} + \ldots \ldots \infty$

$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \ldots \infty$
where co-ratio of $x$ is obtained by dropping co if present and adding co if absent. Thus, sin $\rightarrow$ cos, tan $\rightarrow$ cot, sec $\rightarrow$ cosec.

The sign or is decided from the quadrant in which $n. 90^\circ$ lies.

(iv) Signs of t-ratios in different quadrants (Fig. 2)

(v) $\cos^2 x + \sin^2 x = 1$, $\sec^2 x - \tan^2 x = 1$

(vi) $\sin (x + y) = \sin x \cos y + \cos x \sin y$

(vii) $\cos (x + y) = \cos x \cos y - \sin x \sin y$

(viii) $\tan (x + y) = \tan x + \tan y
\frac{1}{1 - \tan x \tan y}$

(ix) $\sin 2x = 2 \sin x \cos x = \frac{2 \tan x}{1 + \tan^2 x}$

(x) $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

(xi) $\sin 3x = 3 \sin x - 4 \sin^3 x$, $\cos 3x = 4 \cos^3 x - 3 \cos x$

Area of $\Delta ABC = \frac{1}{2} \cdot a \cdot \sin B + \frac{1}{2} \cdot b \cdot \sin C = \frac{1}{2} \cdot c \cdot \sin A$, where $s = \frac{1}{2} (a + b + c)$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where $R$ is the circum-radius of $\Delta ABC$.

$R = \frac{abc}{4A}$

$r = \frac{a}{s}$, where $r$ is the radius of inscribed circle of $\Delta ABC$.

11. De Moivre's Theorem: $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

Euler's Theorem: $\cos \theta + i \sin \theta = e^{i\theta}$.

12. Hyperbolic Functions

\[
\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}
\]

\[
\cosech x = \frac{1}{\sinh x}, \quad \sech x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}
\]

\[
\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad \cosh^2 x - \sinh^2 x = 1
\]

\[
\sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right), \quad \cosh^{-1} x = \log \left( x + \sqrt{x^2 - 1} \right)
\]

\[
\tanh^{-1} x = \frac{1}{2} \log \frac{1 + x}{1 - x}
\]

13. Series

\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \infty
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \infty
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \infty
\]

\[
\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots - \infty, \quad \log (1 - x) = -\left( x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \right)
\]

\[
\tan^{-1} x = \frac{x^3}{3} + \frac{x^5}{5} - \ldots + \infty
\]

\[
\tanh^{-1} x = \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \infty
\]

(Series)

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

(Gregory series)
14. Calculus

(a) Standard Limits

(i) \( \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, \ n \in \mathbb{Q} \)

(ii) \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

(iii) \( \lim_{x \to 0} \frac{\tan x}{x} = 1 \)

(iv) \( \lim_{x \to 0} \frac{1}{x} = 1 \)

(v) \( \lim_{x \to 0} (1 + x)^2 = e \)

(vi) \( \lim_{x \to 0} \frac{a^x - 1}{x} = \log_a a \)

(vii) \( \lim_{x \to 0} x^{1/x} = 1 \)

(ix) If \( f(a) = g(a) = 0 \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \) (L' Hospital's Rule)

(Differentiate the numerator and denominator separately)

(b) Differentiation

(i) \( \frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx} \) (Product Rule)

(ii) \( \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \) (Quotient Rule)

(iii) \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \) (Chain Rule)

(iv) If \( x = f(t), y = g(t) \), then \( \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \)

(v) \( \frac{d}{dx} (e^x) = e^x \)

(vi) \( \frac{d}{dx} (a^x) = a^x \log_a x \)

(ix) \( \frac{d}{dx} (\log_a x) = \frac{1}{x} \log_a x \)

(xi) \( \frac{d}{dx} (\sin x) = \cos x \)

(xii) \( \frac{d}{dx} (\cos x) = -\sin x \)

(xiii) \( \frac{d}{dx} (\tan x) = \sec^2 x \)

(xiv) \( \frac{d}{dx} (\cot x) = -\csc^2 x \)

(xv) \( \frac{d}{dx} (\sec x) = \sec x \tan x \)

(xvi) \( \frac{d}{dx} (\csc x) = -\csc x \cot x \)

(xvii) \( \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \)

(xviii) \( \frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \)

(c) Integration

(i) \( \int x^n \, dx = \frac{x^{n+1}}{n+1} \ (n \neq -1) \)

(ii) \( \int \frac{1}{x} \, dx = \log_a x \)

(iii) \( \int e^x \, dx = e^x \)

(iv) \( \int a^x \, dx = \frac{a^x}{\log_a a} \)

(v) \( \int \sin x \, dx = -\cos x \)

(vi) \( \int \cos x \, dx = \sin x \)

(vii) \( \int \tan x \, dx = -\log \cos x \)

(viii) \( \int \sec x \, dx = \tan x \)

(ix) \( \int \sec^2 x \, dx = \tan x \)

(x) \( \int \csc x \, dx = \log \csc x \)

(xi) \( \int \csc x \cot x \, dx = -\csc x \)

(xii) \( \int \csc^2 x \, dx = -\cot x \)

(xiii) \( \int \sec x \, dx = \log (\sec x + \cot x) = \log \frac{x}{2} \)

(xiv) \( \int \csc x \, dx = \log (\csc x - \cot x) = \log \frac{x}{2} \)

(xv) \( \int \frac{dx}{a^2 + x^2} = \frac{1}{2a} \tan^{-1} \frac{x}{a} \)

(xvi) \( \int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{2a} \log \frac{x + a}{a} \)

(xvii) \( \int \frac{dx}{\sqrt{a^2 + x^2}} = \frac{1}{2a} \log x + a \)

(xviii) \( \int \frac{dx}{\sqrt{x^2 - a^2}} = \frac{1}{2a} \log \frac{x - a}{a} \)

(xix) \( \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \)

(xx) \( \int \sqrt{a^2 + x^2} \, dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a} \)

(xxi) \( \int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cos^{-1} \frac{x}{a} \)

(xxii) \( \int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a} \)

(xxiii) \( \int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cos^{-1} \frac{x}{a} \)
\[ \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left( \alpha \sin bx - \beta \cos bx \right) \]
\[ \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left( \alpha \cos bx + \beta \sin bx \right) \]
\[ \int \sinh x \, dx = \cosh x \]
\[ \int \cosh x \, dx = \sinh x \]
\[ \int \tanh x \, dx = \log \cosh x \]
\[ \int \coth x \, dx = \log \sinh x \]
\[ \int \text{sech}^2 x \, dx = \tanh x \]
\[ \int \text{cosech}^2 x \, dx = -\coth x \]
\[ \int_0^b f(x) \, dx = -\int_b^a f(x) \, dx \]
\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx, \quad a < c < b \]
\[ \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \]
\[ \int_a^b f(x) \, dx = \int_a^b f(a+b-x) \, dx \]
\[ \int_a^a f(x) \, dx = \begin{cases} 0, & \text{if } f \text{ is an odd function} \\
\frac{1}{2} \int_0^a f(x) \, dx, & \text{if } f \text{ is an even function} \end{cases} \]
\[ \int_0^{2a} f(x) \, dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\
\frac{1}{2} \int_0^{2a} f(x) \, dx, & \text{if } f(2a-x) = f(x) \end{cases} \]
\[ \int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{(n-1)(n-3)(n-5)\cdots}{2(n-2)(n-4)\cdots} \times \left( \frac{\pi}{2} \right)^{\frac{n}{2}}, & \text{only if } n \text{ is even} \end{cases} \]
\[ \int_0^{\pi/2} \sin^m x \cos^n x \, dx \]
\[ = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots} \times \left( \frac{\pi}{2} \right)^{\frac{m+n}{2}}, \quad \text{only if both } m \text{ and } n \text{ are even} \]
CHAPTER 1

Fourier Series

1.1. PERIODIC FUNCTIONS

A function $f(x)$ which satisfies the relation $f(x + T) = f(x)$ for all $x$ is called a periodic function. The smallest positive number $T$, for which this relation holds, is called the period of $f(x)$.

If $T$ is the period of $f(x)$, then $f(x) = f(x + T) = f(x + 2T) = \ldots = f(x + nT) = \ldots$

Also $f(x) = f(x - T) = f(x - 2T) = \ldots = f(x - nT) = \ldots$

$\therefore f(x) = f(x \pm nT)$, where $n$ is a positive integer.

Thus, $f(x)$ repeats itself after periods of $T$.

For example, $\sin x$, $\cos x$, $\sec x$ and $\cosec x$ are periodic functions with period $2\pi$ while $\tan x$ and $\cot x$ are periodic functions with period $\pi$. The functions $\sin nx$ and $\cos nx$ are periodic with period $\frac{2\pi}{n}$.

The sum of a number of periodic functions is also periodic. If $T_1$ and $T_2$ are the periods of $f(x)$ and $g(x)$, then the period of $a \cdot f(x) + b \cdot g(x)$ is the least common multiple of $T_1$ and $T_2$.

For example, $\cos x$, $\cos 2x$, $\cos 3x$ are periodic functions with periods $2\pi$, $\pi$ and $\frac{2\pi}{3}$ respectively.

$\therefore f(x) = \cos x + \frac{1}{2} \cos 2x + \frac{1}{3} \cos 3x$ is also periodic with period $2\pi$, the L.C.M. of $2\pi$, $\pi$ and $\frac{2\pi}{3}$.

1.2. FOURIER SERIES

Periodic functions are of common occurrence in many physical and engineering problems; for example, in conduction of heat and mechanical vibrations. It is useful to express these functions in a series of sines and cosines. Most of the single valued functions which occur in applied mathematics can be expressed in the form

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

within a desired range of values of the variable. Such a series is known as Fourier Series. Thus, any function $f(x)$ defined in the interval $c_1 \leq x \leq c_2$ can be expressed in the Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0$, $a_n$, $b_n$ $(n = 1, 2, 3, \ldots)$ are constants, called the Fourier co-efficients of $f(x)$. 

3
Note. To determine \( a_n \) and \( b_n \), we shall need the following results: \( m \) and \( n \) are integers.

\[
\int_{c}^{c+2\pi} \sin nx \, dx = \begin{cases} \frac{\sin nx}{n} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}
\]

\[
\int_{c}^{c+2\pi} \cos nx \, dx = \begin{cases} \frac{\cos nx}{n} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}
\]

(i) \[
\int_{c}^{c+2\pi} \sin nx \, dx = 0, \quad \int_{c}^{c+2\pi} \cos nx \, dx = 0, \quad n \neq 0
\]

(ii) \[
\int_{c}^{c+2\pi} \sin mx \cos nx \, dx = \frac{1}{2} \left( \frac{\sin (m+n)x + \sin (m-n)x}{m+n} \right) - \frac{1}{2} \left( \frac{\sin (m+n)x - \sin (m-n)x}{m-n} \right)
\]

(iii) \[
\int_{c}^{c+2\pi} \cos mx \cos nx \, dx = \frac{1}{2} \left( \frac{\cos (m+n)x + \cos (m-n)x}{m+n} \right) - \frac{1}{2} \left( \frac{\cos (m+n)x - \cos (m-n)x}{m-n} \right)
\]

(iv) \[
\int_{c}^{c+2\pi} \sin mx \sin nx \, dx = \frac{1}{2} \left( \frac{\sin (m+n)x - \sin (m-n)x}{m+n} \right) - \frac{1}{2} \left( \frac{\sin (m+n)x + \sin (m-n)x}{m-n} \right)
\]

(v) \[
\int_{c}^{c+2\pi} \cos mx \cos nx \, dx = \frac{1}{2} \left( \frac{\cos (m+n)x + \cos (m-n)x}{m+n} \right) = \frac{\pi}{n}, \quad n \neq 0
\]

(vi) \[
\int_{c}^{c+2\pi} \sin mx \sin nx \, dx = \frac{1}{2} \left( \frac{\sin (m+n)x - \sin (m-n)x}{m+n} \right) = \frac{\pi}{n}, \quad n \neq 0
\]

To integrate the product of two functions, one of which is a positive integral power of \( x \), we apply the generalized rule of integration by parts. If \( u \) and \( v \) are functions of \( x \), then

\[
\int u \, dv = uv - \int v \, du + \int v \, du + \int v \, du + \ldots
\]

where \( u \) and \( v \) are functions of \( x \), i.e., integral of the product of two functions = 1st function \( \times \) integral of 2nd -- go on differentiating 1st, integrating 2nd, signs alternately + and -.

Simplification should be done only when the integration is over.

For example,

\[
\int x^3 e^{-2x} \, dx = \frac{x^4 e^{-2x}}{-2} - \frac{4x^3 e^{-2x}}{8} + \frac{6x^2 e^{-2x}}{24} - \frac{6x e^{-2x}}{24} + \frac{12 e^{-2x}}{48}
\]

\[
= e^{-2x} \left[ \frac{1}{2} x^4 - \frac{3}{4} x^3 + \frac{3}{8} x^2 + \frac{3}{8} x + \frac{12}{32} \right] - \frac{1}{2} e^{-2x} \left[ 4x^3 + 6x^2 + 6x + 3 \right]
\]

\[
\int x^2 \cos nx \, dx = \frac{x^2 \cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2 \cos nx}{n^3}
\]

(c) Even and Odd Functions

A function \( f(x) \) is said to be even if \( f(-x) = f(x) \) e.g., \( x^2, \cos x, \sin^2 x \) are even functions.

The graph of an even function is symmetrical about the y-axis.

A function \( f(x) \) is said to be odd if \( f(-x) = -f(x) \) e.g., \( x^3, \sin x, \tan^2 x \) are odd functions.

The graph of an odd function is symmetrical about the origin.

The product of two even functions or two odd functions is an even function while the product of an even function and an odd function is an odd function.

Also, \( \int_{c}^{c+2\pi} f(x) \, dx = 0 \), when \( f(x) \) is an odd function

and \( \int_{c}^{c+2\pi} f(x) \, dx = 2\int_{0}^{\pi} f(x) \, dx \), when \( f(x) \) is an even function.

1.3 Euler's Formulae

The Fourier series for the function \( f(x) \) in the interval \( c < x < c + 2\pi \) is given by

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos nx + b_n \sin nx \right]
\]

In finding the coefficients \( a_n \), \( a_0 \) and \( b_n \), we assume that the series on the right hand side of (1) is uniformly convergent for \( c < x < c + 2\pi \) and it can be integrated term by term in the given interval.

To find \( a_0 \), integrate both sides of (1) w.r.t. \( x \), between the limits \( c \) to \( c + 2\pi \).

\[
\int_{c}^{c+2\pi} f(x) \, dx = \frac{a_0}{2} \int_{c}^{c+2\pi} dx + \sum_{n=1}^{\infty} \left[ a_n \cos nx \right] \int_{c}^{c+2\pi} dx + \sum_{n=1}^{\infty} b_n \sin nx \int_{c}^{c+2\pi} dx
\]

\[
= \frac{a_0}{2} (2\pi - c) + 0 + 0
\]

[By formula (i) above]

\[
a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \, dx
\]

To find \( a_n \), multiply both sides of (1) by \( \cos nx \) and integrate w.r.t. \( x \), between the limits \( c \) to \( c + 2\pi \).

\[
\int_{c}^{c+2\pi} f(x) \cos nx \, dx = \frac{a_0}{2} \int_{c}^{c+2\pi} \cos nx \, dx + \sum_{n=1}^{\infty} \left[ a_n \cos nx \right] \int_{c}^{c+2\pi} dx + \sum_{n=1}^{\infty} b_n \sin nx \int_{c}^{c+2\pi} \cos nx \, dx
\]

\[
= 0 + a_n \pi + 0
\]

[By formulae (i), (v) and (vi)]

\[
a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cos nx \, dx
\]
To find $b_n$, multiply both sides of (1) by $\sin nx$ and integrate w.r.t. $x$ between the limits $c$ to $c + 2\pi$.

$$
\int_c^{c+2\pi} f(x) \sin nx \, dx = \frac{a_0}{2} \int_c^{c+2\pi} \sin nx \, dx + \int_c^{c+2\pi} \left( \sum_{n=1}^{\infty} a_n \cos nx \right) \sin nx \, dx
$$

$$
= 0 + 0 + \sum_{n=1}^{\infty} b_n \sin nx \sin nx \, dx \quad [\text{By formulae (i), (ii) and (iv)}]
$$

$$
b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx \, dx
$$

Hence $a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \, dx$; $a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx \, dx$; and $b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx \, dx$.

These values of $a_0$, $a_n$ and $b_n$ are called Euler's formulae.

Cor. 1. If $c = 0$, the integral becomes $0 < x < 2\pi$ and the formulae I reduce to

$$
a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx; \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx; \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx
$$

Cor. 2. If $c = -\pi$, the interval becomes $-\pi < x < \pi$, and the formulae I reduce to

$$
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx; \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx; \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
$$

Cor. 3. When $f(x)$ is an odd function $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0$.

Since $\cos nx$ is an even function, therefore, $f(x) \cos nx$ is an odd function.

$$
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0
$$

Since $\sin x$ is an odd function, therefore, $f(x) \sin nx$ is an even function.

$$
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 2 \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx
$$

Hence, if a periodic function $f(x)$ is odd, its Fourier expansion contains only sine terms,

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \]

When $f(x)$ is an even function $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = 2 \frac{1}{\pi} \int_0^{\pi} f(x) \, dx$

Since $\cos nx$ is an even function, therefore, $f(x) \cos nx$ is an even function.

$$
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx
$$

Since $\sin x$ is an odd function, therefore, $f(x) \sin x$ is an odd function.

$$
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x \, dx = 0
$$

Hence, if a periodic function $f(x)$ is even, its Fourier expansion contains only cosine terms,

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad \text{where} \quad a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx \quad \text{and} \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx
\]

### ILLUSTRATIVE EXAMPLES

**Example 1.** Obtain the Fourier series to represent $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 < x < 2\pi$.

Hence obtain the following relations:

\[(i) \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \]

\[(ii) \quad \frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{9^2} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \]

\[(iii) \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} \]

**Sol.** Let $f(x) = \frac{1}{4} (\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have

$$
a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) \, dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x)^2 \, dx = \frac{1}{4\pi} \left[ (\pi - x)^3 \right]_0^\pi = -\frac{1}{12\pi} [\pi^3 - 3\pi^3] = \frac{\pi^2}{6}
$$

$$
a_n = \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x)^2 \cos nx \, dx
$$

$$
= \frac{1}{4\pi} \left[ (\pi - x)^2 \sin nx \right]_0^\pi - \left[ (\pi - x) \left( -(2\pi - x) \right) \frac{\sin nx}{n^2} + 2 \left( \frac{\sin nx}{n^3} \right) \right]_0^\pi
$$

$$
= \frac{1}{4\pi} \left[ 0 + 2\pi \cos 2\pi \frac{1}{n^2} + 0 - (0 - 2\pi \cos 0 + 0) \right] = \frac{1}{4\pi} \left[ 2\pi + 2\pi \right] = \frac{1}{n^2}
$$

$$
b_n = \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x)^2 \sin nx \, dx
$$

$$
= \frac{1}{4\pi} \left[ (\pi - x)^2 \left( \cos nx \right) \right]_0^\pi - \left[ (\pi - x) \left( -(2\pi - x) \right) \frac{\cos nx}{n^3} + 2 \left( \frac{\cos nx}{n^3} \right) \right]_0^\pi
$$

$$
= \frac{1}{4\pi} \left[ \pi^2 \cos 2\pi \frac{1}{n^3} - 0 + 2\cos 2\pi \frac{1}{n^3} \right] = \frac{1}{n^3} (\pi - 0 + 2\cos 0)
$$
\[
\frac{1}{4\pi} \left[ \frac{\pi^2}{n} \frac{2}{n^3} - \frac{\pi^2}{n} \frac{2}{n^3} \right] = 0
\]

\[
\therefore f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \left( \frac{\pi^2}{n} \frac{2}{n^3} + \frac{\cos x}{n^2} \frac{2}{n^3} + \frac{\cos 3x}{n^2} + \ldots \right)
\]

**Deductions**

(i) Putting \( x = 0 \) in equation (1), we get

\[
f(0) = \frac{\pi^2}{12} \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \right)
\]

\[
\Rightarrow \frac{\pi^2}{12} \left[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \right] = \frac{\pi^2}{6}
\]

\[
\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}
\]...

(ii) Putting \( x = \pi \) in equation (1), we get

\[
f(\pi) = \frac{\pi^2}{12} \left[ \frac{-1}{1^2} + \frac{1}{2^2} + \frac{-1}{3^2} + \frac{1}{4^2} + \ldots \right]
\]

\[
\Rightarrow 0 = \frac{\pi^2}{12} \left( \frac{-1}{1^2} + \frac{1}{2^2} + \frac{-1}{3^2} + \frac{1}{4^2} + \ldots \right)
\]

\[
\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{-\pi^2}{12}
\]...

(iii) Adding (2) and (3), we get

\[
2 \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \right) = \frac{\pi^2}{6} + \frac{\pi^2}{6}
\]

\[
\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{1}{2} \left( \frac{\pi^2}{4} \right) = \frac{\pi^2}{8}
\]

**Example 2.** Expand \( f(x) = \sin x \), \( 0 < x < 2\pi \) as a Fourier series.


**Sol.** Let \( f(x) = \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \)

By Euler's formulae, we have \( a_0 = \frac{1}{\pi} \int_{0}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{0}^{\pi} \sin x \, dx \)

\[
= \frac{1}{\pi} \left[ x \sin x - \cos x \right]_{0}^{\pi} = \frac{1}{\pi} [2\pi - 0] = 2
\]

\[a_n = \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{0}^{\pi} \sin x \cos nx \, dx = \frac{1}{\pi} \int_{0}^{\pi} \sin x (2 \cos nx \sin x) \, dx
\]

\[= \frac{1}{2 \pi} \int_{0}^{\pi} x \sin (n+1)x - \sin (n-1)x \, dx
\]

\[= \frac{1}{2 \pi} \left[ -\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1} - \frac{1}{n+1} \sin (n+1)x + \frac{1}{n-1} \sin (n-1)x \right]_{0}^{\pi}
\]

\[= \frac{1}{2 \pi} \left[ \frac{2 \pi}{n+1} \frac{\cos (n+1)\pi}{n+1} - \frac{\cos (n-1)\pi}{n-1} - \frac{1}{n+1} \sin (n+1)\pi + \frac{1}{n-1} \sin (n-1)\pi \right]
\]

\[= \frac{1}{2 \pi} \left[ \frac{2 \pi}{n+1} \frac{1}{n+1} + \frac{2 \pi}{n-1} \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n-1} \right] = \frac{1}{2 \pi} \frac{1}{n+1} - \frac{1}{n-1}, n \neq 1
\]

When \( n = 1 \), we have \( a_1 = \frac{1}{\pi} \int_{0}^{\pi} x \sin x \cos x \, dx = \frac{1}{2 \pi} \int_{0}^{\pi} x \sin 2x \, dx
\]

\[= \frac{1}{2 \pi} \left[ -\frac{\sin 2x}{4} \right]_{0}^{\pi} = \frac{1}{2 \pi} \left[ -\pi \right] = -\frac{1}{2}
\]

\[b_n = \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{0}^{\pi} x \sin x \sin nx \, dx
\]

\[= \frac{1}{\pi} \int_{0}^{\pi} x \sin nx \sin x \, dx
\]

\[= \frac{1}{\pi} \left[ \frac{\sin (n+1)x - \sin (n-1)x}{n+1} - \frac{\sin (n+1)x + \sin (n-1)x}{n-1} - \frac{1}{n+1} \cos (n+1)x + \frac{1}{n-1} \cos (n-1)x \right]_{0}^{\pi}
\]

\[= \frac{1}{\pi} \left[ \frac{\cos (n+1)\pi}{(n+1)^2} + \frac{\cos (n-1)\pi}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2} \right] = 0, n \neq 1
\]

When \( n = 1 \), we have \( b_1 = \frac{1}{\pi} \int_{0}^{\pi} x \sin x \sin x \, dx = \frac{1}{2 \pi} \int_{0}^{\pi} x (1 - \cos 2x) \, dx
\]

\[= \frac{1}{2 \pi} \left[ \frac{x - \frac{x}{2} \cos 2x}{2} \right]_{0}^{\pi} = \frac{1}{2 \pi} \left[ \frac{\pi}{2} \right] = \frac{\pi}{4}
\]

\[= \frac{1}{2 \pi} \left[ 2\pi(2) - 4 \pi^2 + \frac{2}{4} - \frac{1}{4} \right] = \frac{1}{2 \pi} (2 \pi^2) = \pi
\]

\[
\therefore f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx
\]

\[= -\frac{1}{2} \cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1} \cos nx + 0
\]

\[= -1 + \pi x - \frac{1}{2} \cos x + \frac{2}{2} \cos 2x + \frac{2}{3} \cos 3x + \ldots
\]
Example 3. Find a Fourier series to represent \( x - x^2 \) from \( x = -\pi \) to \( x = \pi \). Hence show that

\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.
\]

(V.T.U. 2006; K.U.K. 2009; Madras 2006)

Sol. Let \( x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \)

By Euler's formulae, we have

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \, dx = \frac{1}{\pi} \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}.
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx \, dx = \frac{1}{\pi} \left[ \frac{\cos nx}{n^2} \right]_{-\pi}^{\pi} - 2 \frac{\sin nx}{n^3} \left(1 - \frac{2n}{\pi}\right) = \frac{4}{\pi} \frac{(-1)^n}{n^2}.
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \sin nx \, dx = \frac{1}{\pi} \left[ \frac{\cos nx}{n^2} \right]_{-\pi}^{\pi} - 2 \frac{\sin nx}{n^3} \left(1 - \frac{2n}{\pi}\right) = \frac{4}{\pi} \frac{(-1)^n}{n^2} \cdot \cos \frac{\pi n}{2}.
\]

Then

\[
a_n = \frac{4}{\pi} \frac{(-1)^n}{n^2} \cdot \cos \frac{\pi n}{2} = \frac{4}{\pi} \frac{(-1)^n}{n^2}.
\]

Puttig \( x = \pi \) in (1), we get

\[
\pi^2 = 4 \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \right) \Rightarrow \frac{2\pi^2}{3} = 4 \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \right)
\]

Result (i)

Example 4. Obtain the Fourier series for the function \( f(x) = x^2 \), \( -\pi < x < \pi \). Hence show that

\[
(\text{PTU} \ 2005, \ \text{BPTU} \ 2006)
\]

\[
a_0 = \frac{\int_{-\pi}^{\pi} f(x) \, dx}{\pi} = \frac{\int_{-\pi}^{\pi} x^2 \, dx}{\pi^2} = \frac{2}{\pi^2} \int_{0}^{\pi} x^2 \, dx = \frac{2}{\pi^2} \cdot \frac{\pi^3}{3} = \frac{2}{3} \pi^2
\]

\[
a_n = \frac{\int_{0}^{\pi} f(x) \cos nx \, dx}{\pi} = \frac{\int_{0}^{\pi} x^2 \cos nx \, dx}{\pi^2} = \frac{2}{\pi^2} \int_{0}^{\pi} x^2 \cos nx \, dx = 4 \left( \frac{1}{n^2} \right)
\]

whereby

\[
\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{\pi^2}{6}
\]

Adding (i) and (ii), we get

\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.
\]

Result (iii)

Example 5. Obtain the Fourier series for \( f(x) = e^{-x} \) in the interval \( 0 < x < 2\pi \)

(S.V.T.U. 2007)

Sol. Let

\[
f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx
\]
Then
\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} \, dx = \frac{1}{\pi} \left[ -e^{-x} \right]_{-\pi}^{\pi} = 1 - e^{-2\pi} \]
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} \cos nx \, dx \]
\[ = \frac{1}{\pi} \left[ \frac{e^{-x}}{1 + n^2} (-\sin nx - n \cos nx) \right]_{-\pi}^{\pi} = \frac{1 - e^{-2\pi}}{\pi n (1 + n^2)} \]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} \sin nx \, dx \]
\[ = \frac{1}{\pi} \left[ \frac{e^{-x}}{1 + n^2} (-\sin nx + n \cos nx) \right]_{-\pi}^{\pi} = \frac{1 - e^{-2\pi}}{\pi n (1 + n^2)} \]

\[ e^{-x} = \frac{1 - e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty} \frac{1 - e^{-2\pi}}{n^2} \sin nx \]
\[ = \frac{1 - e^{-2\pi}}{\pi} \left[ \frac{\sin x}{2} + \frac{1}{10} \cos 3x + \cdots \right] \]

Example 6. Find the Fourier series to represent \( e^{-x} \) in the interval \(-\pi < x < \pi\). Hence derive series for \( \frac{\pi}{\sinh \pi} \).

Sol. Let \( f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \)

Then
\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} \, dx = \frac{1}{\pi} \left[ e^{-x} - e^{-\pi} \right] = 2 \sinh \pi \ln(2) \]
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} \cos nx \, dx \]
\[ = \frac{1}{\pi} \left[ \frac{e^{-x}}{a^2 + n^2} (a \cos nx + n \sin nx) \right] = \frac{1}{\pi a^2 + n^2} \left[ a e^{-x} \cos nx - a e^{-x} \cos nx \right] \]
\[ = \frac{a \cos nx (e^{-x} - e^{-\pi})}{\pi a^2 + n^2} \]

Example 8. Expand the function \( f(x) = x \sin x \) as a Fourier series in the interval \(-\pi \leq x \leq \pi\).

(U.P.T.U., 2008)

Deduce that
\[ \frac{1}{3} - \frac{1}{13} + \frac{1}{35} - \frac{1}{79} + \cdots = \frac{\pi - 2}{4} \]

(U.P.T.U., 2008)
Sol. Since \( \sin x \) is an even function of \( x \), \( b_n = 0 \)

Let 
\[
\phi(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx
\]

Then 
\[
a_0 = \frac{\pi}{2} \int_0^\pi x \sin x \, dx = 2 \left[ x(-\cos x) - 1 \right]_{-\pi}^{\pi} = 2 \left( -\pi \cos \pi \right) = 2
\]

\[
a_n = \frac{\pi}{2} \int_0^\pi \sin nx \, dx = \frac{\pi}{2} \left[ n \sin (n+1)x - (n-1) \sin (n-1)x \right]_{-\pi}^{\pi} = \frac{\pi}{2} \left( \sin (n+1)\pi - \sin (n-1)\pi \right)
\]

\[
\begin{align*}
&= \frac{\pi}{2} \left[ \frac{\cos (n+1)x + \cos (n-1)x}{n+1} - \frac{\cos (n+1)x - \cos (n-1)x}{n-1} \right]_{-\pi}^{\pi} \\
&= \frac{\pi}{2} \left[ \frac{\cos (n+1)\pi + \cos (n-1)\pi}{n+1} - \frac{\cos (n+1)\pi - \cos (n-1)\pi}{n-1} \right] \\
&= \frac{\cos (n+1)\pi - \cos (n-1)\pi}{n+1 - n-1} \\
&= \frac{\cos (n+1)\pi - \cos (n-1)\pi}{2}
\end{align*}
\]

When \( n \) is odd, \( n+1, n-1 \) and \( n+1 \) are even

\[a_n = \frac{1}{n+1} - \frac{1}{n-1} = \frac{2}{n^2 - 1}
\]

When \( n \) is even, \( n-1 \) and \( n+1 \) are odd

\[a_n = -\frac{1}{n+1} + \frac{1}{n-1} = \frac{-2}{n^2 - 1}
\]

When \( n = 1 \), we have \( a_1 = \frac{2}{\pi} \int_0^\pi x \sin x \, dx = \frac{1}{\pi} \int_0^\pi x \sin 2x \, dx
\]

\[= \frac{\pi}{2} \left[ \frac{\cos 2x}{2} - \frac{\sin 2x}{4} \right]_{-\pi}^{\pi} = \frac{\pi}{2} \left( \frac{\cos 2\pi}{2} - \frac{\sin 2\pi}{4} \right) = -\frac{1}{2}
\]

\[x \sin x = \frac{1}{2} \cos x - \frac{1}{2} \cos 2x + \frac{1}{2} \cos 3x + \cdots
\]

Putting \( x = \frac{\pi}{2} \) we get \( \frac{\pi}{2} = 1 - 2 \left( \frac{1}{2^2 - 1} - \frac{1}{3^2 - 1} + \frac{1}{4^2 - 1} - \cdots \right)
\]

\[\Rightarrow \frac{\pi}{2} = 1 - 2 \left( \frac{1 - 2}{2^2 - 1} + \frac{1 - 2}{3^2 - 1} - \cdots \right) \Rightarrow \frac{\pi}{1.3} = 1.3 = 1.3 = 3.5 = 5.7 - \cdots
\]

Example 9. Show that for \( -\pi < x < \pi \),

\[
\sin ax = \frac{2 \sin \pi}{\pi} \left( \frac{\sin x}{n^2 - a^2} - \frac{2 \sin 2x}{2^2 - a^2} + \frac{3 \sin 3x}{3^2 - a^2} - \cdots \right)
\]

Sol. Since \( \sin ax \) is an odd function of \( x \), \( a_0 = 0 \) and \( a_n = 0 \).

Let \( \sin ax = \sum_{n=1} b_n \sin nx \)

Example 10. Obtain Fourier series for the function \( f(x) \) given by

\[
f(x) = 1 + \frac{2x}{\pi}, \quad -\pi \leq x \leq 0
\]

\[
= 1 - \frac{2x}{\pi}, \quad 0 \leq x \leq \pi.
\]

Hence deduce that \( \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} \)

Sol. When \( -\pi \leq x \leq 0 \), \( 0 \leq x \leq \pi \)

\[
\frac{f(x) - f(-x)}{2} = 1 + \frac{2x}{\pi}
\]

When \( 0 \leq x \leq \pi \), \( -\pi \leq x \leq 0 \)

\[
f(x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = f(x)
\]

\( f(x) \) is an even function of \( x \) in \( [-\pi, \pi] \). This is also clear from its graph which is symmetrical above the y-axis.

\( b_n = 0 \)

Let \( f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx \)

then

\[
a_0 = \frac{2}{\pi} \int_0^\pi f(x) \, dx = \frac{2}{\pi} \int_0^\pi \left( 1 - \frac{2x}{\pi} \right) \, dx = \frac{2}{\pi} \left[ x - \frac{x^3}{3} \right]_0^\pi = 0
\]

\[
a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^\pi \left( 1 - \frac{2x}{\pi} \right) \cos nx \, dx
\]

\[
= \frac{2}{\pi} \left[ \frac{1}{\pi} \left( -\frac{2}{\pi} \right) \sin nx - \left( \frac{2}{\pi} \right) \cos nx \right]_0^\pi
\]
\[ f(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} \]

Putting \( x = 0 \), we get \( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8} \) since \( f(0) = 1 \).

Example 11. Show that for \(-\pi \leq x \leq \pi\),

\[ \cos cx = \frac{\sin c \pi}{\pi} \left[ 1 - \frac{2c \cos x + 2c \cos 2x}{c^2 - 1^2} + \frac{2c \cos 3x - 2c \cos 2x}{c^2 - 2^2} - \ldots \right] \]

where \( c \) is non-integral. Hence deduce that

\[ \pi \cosec (cn) = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{n+c} + \frac{1}{n+1-c} \right] \quad \text{(M.D.U. Dec. 2011)} \]

Sol. Since \( \cos cx \) is an even function of \( x \), \( b_n = 0 \)

\[ a_0 = \frac{1}{\pi} \int_0^\pi \cos cx \, dx = \frac{\sin cx}{c} \bigg|_0^\pi = 2 \sin \pi c, \quad \text{since } c \text{ is non-integral, } \sin \pi c \neq 0 \]

Then

\[ a_n = \frac{2}{\pi} \int_0^\pi \cos cx \cos nx \, dx = \frac{1}{\pi} \int_0^\pi \left[ \cos (n+c)x + \cos (n-c)x \right] \, dx \]

Also,

\[ a_n = \frac{2}{\pi} \int_0^\pi \cos cx \cos nx \, dx = \frac{1}{\pi} \int_0^\pi \left[ \sin (n+c)x \frac{\sin (n-c)x}{n+c} \right] \bigg|_0^\pi \]

Hence

\[ \pi \cosec (cn) = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{n+c} + \frac{1}{n+1-c} \right] \]

| EXERCISE 1.1 |

1. Expand in a Fourier series the function \( f(x) = x \) in the interval \( 0 < x < 2\pi \).
2. Express \( f(x) = \frac{1}{2} (n-x) \) in a Fourier series in the interval \( 0 < x < 2\pi \).
3. Find the Fourier series for the function \( f(x) = x + x^2 \) in the interval \(-\pi < x < \pi\).

Hence show that

(i) \( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{12} \) \hspace{1cm} (ii) \( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6} \)
4. Prove that for all values of \( x \) between \( -\pi \) and \( \pi \), \[ x = 2 \left( \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \cdots \right) \] (M.D.U. May 2011)

5. Obtain the Fourier series to represent \( e^x \) in the interval \( 0 < x < 2\pi \).

6. Find the Fourier series to represent \( e^{-x} \) in the interval \( -\pi < x < \pi \). (M.D.U. May 2012)

7. Find the Fourier series to represent the function \( f(x) = |\sin x| \), \( -\pi < x < \pi \). (M.D.U. 2007)

8. Expand \( f(x) = |\cos x| \) as a Fourier series in the interval \(-\pi < x < \pi \). (M.D.U. 2005, Dec. 2010; K.U.K. 2009)

9. Prove that in the interval \(-\pi < x < \pi\), \( x \cos x = -\frac{1}{2} \sin x + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-1} \sin nx \). (U.P.T.U. 2008)

10. Prove that for \(-\pi < x < \pi\), \[ \frac{x^3 - x^5}{12} = \frac{\sin x}{x} \frac{x^2 - \sin 2x}{2^3} \sin \frac{3x}{3^2} \sin \frac{4x}{4^2} + \cdots \]

11. (a) Obtain a Fourier expansion for \( \sqrt{1 - \cos x} \) in the interval \(-\pi < x < \pi \).

   \[ \text{Hint: For all integral values of } n, \cos \left( n + \frac{1}{2} \right) x = \cos (2n + 1) \frac{x}{2} \cos \left( n - \frac{1}{2} \right) x \]

   (b) Obtain a Fourier series for \( \sqrt{1 - \cos x} \) in the interval \(0, 2\pi) and hence find the value of \[ \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \] (Bombay 2006; J.N.T.U. 2006)

12. Express \( f(x) = |\cos x| \) as \( -\pi < x < \pi \), where \( w \) is a fraction, as a Fourier series. Hence prove that \[ \cot \frac{x}{2} = \frac{1}{2} \frac{2}{1} - \frac{1}{3} \frac{2}{3} + \frac{2}{5} - \cdots \]

13. Find the Fourier series for \( f(x) = x \), \(-\pi < x < \pi \) when \( f(x) = \pi - x \), \( -\pi < x < 0 \) \( = x \), \( 0 < x < \pi \).

14. Obtain a Fourier series to represent \( e^{-x} \) from \( x = -\pi \) to \( x = \pi \). Hence derive series for \( \frac{x}{\sinh x} \). (M.D.U. May 2009)

15. Prove that in the range \(-\pi < x < \pi\), \( \cosh ax = \frac{2a}{\pi} \sinh ax \left[ \frac{1}{3a^2} + \sum_{n=1}^\infty \frac{(-1)^n}{n^2-a^2} \cos nx \right] \)

16. Given \( f(x) = \begin{cases} x + 1 & \text{for} \ -\pi < x < 0 \\ 1 & \text{for} \ 0 \leq x \leq \pi \end{cases} \)

   Is the function even or odd? Find the Fourier series for \( f(x) \) and deduce the value of \[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \]

   \[ \text{Answers} \]

1. \( f(x) = \pi - 2 \sum_{n=1}^\infty \frac{\sin nx}{n} \)

2. \( f(x) = \sum_{n=1}^\infty \frac{\sin nx}{n} \)

14. DIRICHLET'S CONDITIONS

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions (after Dirichlet, a German mathematician). All the functions that normally arise in engineering problems satisfy these conditions and hence they can be expressed as a Fourier series.

Any function \( f(x) \) can be expressed as a Fourier series \[ \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx + \sum_{n=1}^\infty b_n \sin nx \]

where \( a_0, a_n, b_n \) are constants, provided

(i) \( f(x) \) is periodic, single valued and finite.

(ii) \( f(x) \) has a finite number of finite discontinuities in any one period.

(iii) \( f(x) \) has a finite number of maxima and minima.

When these conditions are satisfied, the Fourier series converges to \( f(x) \) at every point of continuity. At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left.
where \( f(x + 0) \) and \( f(x - 0) \) denote the limit on the right and the limit on the left respectively.

### 1.5. FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

In 1.3, we derived Euler's formulae for \( a_n \), \( b_n \), on the assumption that \( f(x) \) is continuous in \((c, c + 2\pi)\). However, if \( f(x) \) has finitely many points of finite discontinuity, then it can be expressed as a Fourier series. The integrals for \( a_n \), \( b_n \) are to be evaluated by breaking up the range of integration.

Let \( f(x) \) be defined by \( f(x) = f_1(x) \), \( c < x < x_0 \)
\[ f_1(x) = f_2(x) \], \( x_0 < x < c + 2\pi \)
where \( x_0 \) is the point of finite discontinuity in the interval \((c, c + 2\pi)\).

The values of \( a_n \), \( b_n \) are given by

\[
\begin{align*}
a_n &= \frac{1}{\pi} \int_{c}^{x_0} f_1(x) \cos nx \, dx + \int_{x_0}^{c+2\pi} f_2(x) \cos nx \, dx, \\
b_n &= \frac{1}{\pi} \int_{c}^{x_0} f_1(x) \sin nx \, dx + \int_{x_0}^{c+2\pi} f_2(x) \sin nx \, dx.
\end{align*}
\]

At \( x = x_0 \), there is a finite jump in the graph of the function. Both the limits \( f(x_0 - 0) \) and \( f(x_0 + 0) \) exist but are unequal. The sum of the Fourier series is \( \frac{1}{2} [f(x_0 - 0) + f(x_0 + 0)] = \frac{1}{2} (AH + AC) = AM \), where M is the mid-point of BC.

### ILLUSTRATIVE EXAMPLES

**Example 1.** Find the Fourier series to represent the function \( f(x) \) given by

\[ f(x) = x \quad \text{for} \quad 0 \leq x \leq \pi \]

and

\[ = 2\pi - x \quad \text{for} \quad \pi \leq x \leq 2\pi. \]  

(B.P.T.U. 2005 S)

Deduce that \( \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \ldots = \frac{\pi^2}{8} \).

**Sol.** Let \( f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \)

Then

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
\]

Putting \( x = 0 \), we get \( 0 = \frac{\pi}{2} - \frac{4}{\pi} \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right) \Rightarrow \pi^2 = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \)

(Bombay, 2005 S; P.T.U. 2005; M.D.U. 2005)

**Example 2.** If \( f(x) = \begin{cases} 0, & -n \leq x \leq 0 \\ \sin x, & 0 \leq x \leq \pi, \end{cases} \)
prove that \( f(x) = \frac{1}{n^2} \left( \sin x + \frac{\cos 2nx}{4n^2 - 1} \right) \)

(Bombay, 2005 S; P.T.U. 2005; M.D.U. 2005)

Hence show that

\[
(\text{i}) \quad \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots = \frac{1}{2} \\
(\text{ii}) \quad \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots = \frac{\pi - 2}{4}
\]

(M.D.U. 2005)

**Sol.** Let \( f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \)

Then

\[
a_n = \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx
\]

Putting \( x = 0 \), we get \( 0 = \frac{\pi}{2} - \frac{4}{\pi} \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right) \Rightarrow \pi^2 = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \)

(Bombay, 2005 S; P.T.U. 2005; M.D.U. 2005)
Then
\[
\alpha_n = \frac{1}{\pi} \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} f(x) \sin nx \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} \sin x \, dx \right] = \frac{2}{\pi}
\]
and
\[
\beta_n = \frac{1}{\pi} \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} f(x) \cos nx \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} \sin x \cos nx \, dx \right]
\]

\[
= \frac{1}{2\pi} \int_{0}^{\pi} 2 \cos nx \sin x \, dx = \frac{1}{2\pi} \int_{0}^{\pi} [\sin ((n+1)x) - \sin ((n-1)x)] \, dx
\]

\[
= \frac{1}{2\pi} \left[ \frac{\cos ((n+1)x)}{n+1} - \frac{\cos ((n-1)x)}{n-1} \right] \bigg|_{0}^{\pi} = \frac{1}{2\pi} \left[ \frac{\cos (n+1)\pi}{n+1} - \frac{\cos (n-1)\pi}{n-1} \right]
\]

\[
= \frac{1}{2\pi} \left[ \frac{\cos (n+1)\pi + \cos (n-1)\pi}{n+1} + \frac{1}{n+1} - \frac{1}{n-1} \right]
\]

\[
= \frac{1}{2\pi} \left[ \frac{\cos (n+1)\pi}{n+1} + \frac{1}{n+1} - \frac{1}{n-1} \right], \text{ when } n \text{ is odd}
\]

\[
= \frac{1}{2\pi} \left[ \frac{1}{n+1} + \frac{1}{n-1} \right], \text{ when } n \text{ is even}
\]

When \( n = 1 \), we have
\[
\alpha_1 = \frac{1}{\pi} \int_{0}^{\pi} \sin x \, dx = \frac{2}{\pi}
\]
and
\[
\beta_1 = \frac{1}{\pi} \int_{0}^{\pi} \cos x \, dx = 0
\]

Thus
\[
f(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \left[ \frac{\cos 2\pi}{n} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \right]
\]

Putting \( x = \frac{\pi}{2} \) in (1), we have
\[
\frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{3}{\pi} + \frac{1}{\pi} + \frac{1}{\pi}
\]

Also deduce that \( \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \)

Exercise 1.2

1. Find the Fourier series to represent the function
\[
f(x) = k \quad \text{when} -\pi < x < 0
\]
\[
= h \quad \text{when} 0 < x < \pi
\]

Also deduce that \( \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \)

(b) Develop \( f(x) \) in a Fourier series in the interval \( (-\pi, \pi) \) if \( f(x) = 0 \) when \( -\pi < x < 0 \)

Deduce that sum of the Gregory series \( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4} \).

(M.U.D. Dec. 2011)

3. Find the Fourier series expansion for \( f(x) \), if \( f(x) = -\pi \), \(-\pi < x < 0\)

Deduce that \( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8} \).

(M.U.D. Dec. 2009)

5. Find the Fourier series of \( f(x) = \frac{1}{x^2} \), \( 0 < x < \pi \)

which is assumed to be periodic with period \( 2\pi \).
6. Find the Fourier series of the following function
\[ f(x) = \begin{cases} x & 0 \leq x \leq \pi \\ -x & -\pi \leq x \leq 0 \end{cases} \] 
where \( l \) is the maximum current and the period is \( 2\pi \). Express \( f(x) \) as a Fourier series.

7. An alternating current after passing through a rectifier has the form
\[ i = l_0 \sin x \quad \text{for} \quad 0 \leq x \leq \pi \]
\[ = 0 \quad \text{for} \quad \pi \leq x \leq 2\pi \]
where \( l_0 \) is the maximum current and the period is \( 2\pi \). Express \( i \) as a Fourier series.

8. Obtain Fourier series for the function
\[ f(x) = \begin{cases} x & -\pi < x < 0 \\ -x & 0 < x < \pi \end{cases} \]
and hence show that
\[ \frac{1}{l^2} \frac{d^2}{dx^2} \frac{1}{5} + \ldots = \frac{\pi^2}{8}. \]

9. Find the Fourier series for the function
\[ f(x) = \begin{cases} -1 & -\frac{\pi}{2} < x < -\frac{\pi}{2} \\ 0 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases} \]

**Answers**

1. \[ f(x) = \frac{4x}{\pi} \left( \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right) \]
2. \[ f(x) = \frac{1}{2} \sin x + \frac{2}{\pi} \left( \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right) \]
3. \[ f(x) = 0 = -\frac{1}{2} \left( \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \ldots \right) \]
4. \[ f(x) = \frac{3}{4} \sin x + \frac{2}{\pi} \left( \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right) \]
5. \[ f(x) = \frac{1}{2} \left( \frac{1}{4} - \frac{2}{3} \right) \sin x + \frac{1}{2} \sin 2x + \frac{4}{3} - \frac{\pi^2}{3} \left( \frac{\sin x}{4} + \frac{\sin 3x}{4} + \ldots \right) \]
6. \[ f(x) = 2 \left( \frac{4}{\pi} \right) \sin x - \pi \sin 2x + 2 \left( \frac{4}{3} - \frac{\pi}{2} \right) \sin 3x - \frac{\pi}{2} \sin 4x + \ldots \]
7. \[ i = \frac{1}{\pi} \sin x + \frac{4}{\pi} \left( \frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \ldots \right) \]
8. \[ f(x) = -\frac{4}{\pi} \left( \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots \right) \]
9. \[ f(x) = \frac{2}{\pi} \left( \frac{\sin 3x}{3} - \ldots \right) \]

### Fourier Series

#### 1.6 Change of Interval

In many engineering problems, it is desired to expand a function in a Fourier series over an interval of length \( 2l \) and not \( 2\pi \). In order to apply the foregoing theory, this interval must be transformed into an interval of length \( 2\pi \). This can be achieved by a transformation of the variable.

Consider a periodic function \( f(x) \) defined in the interval \( c < x < c + 2l \). To change the interval into one of length \( 2\pi \), we put
\[ x = \frac{\pi}{l} z \quad \text{or} \quad z = \frac{\pi x}{l} \]
so that when \( x = c \), \( z = \frac{\pi c}{l} = d \) (say)

and when \( x = c + 2l \), \( z = \frac{\pi (c + 2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi \).

Thus the function \( f(x) \) of period \( 2l \) in \( (c, c + 2l) \) is transformed to the function
\[ f\left( \frac{\pi x}{l} \right) = F(z) \]
say, of period \( 2\pi \) in \( (d, d + 2\pi) \) and the last function can be expressed as the Fourier series
\[ F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \]

... (1)

where \( a_n = \frac{1}{\pi} \int_{c}^{c+2l} f(x) \cos \frac{n\pi x}{l} \, dx \) and \( b_n = \frac{1}{\pi} \int_{c}^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx \)

Now making the inverse substitution \( z = \frac{\pi x}{l} \), \( dx = \frac{\pi}{l} \, dz \)

When \( x = d \), \( z = c \) and when \( x = d + 2\pi \), \( z = c + 2l \).

The expression (1) becomes
\[ F(z) = \int_{d}^{d+2\pi} f(z) \, dz = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \]

and the coefficients \( a_n, b_n \) from (2) reduce to
\[ a_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \cos \frac{n\pi x}{l} \, dx \] and \[ b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx \]

Hence the Fourier series for \( f(x) \) in the interval \( c < x < c + 2l \) is given by
\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \]

where \( a_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \, dx \) and \( b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx \).

**Cor. 1.** If we put \( c = 0 \), the interval becomes \( 0 < x < 2l \) and the above results reduce to
\[ a_n = \frac{1}{l} \int_{0}^{2l} f(x) \cos \frac{n\pi x}{l} \, dx \] and \[ a_n = \frac{1}{l} \int_{0}^{2l} f(x) \sin \frac{n\pi x}{l} \, dx \]

**Cor. 2.** If we put \( c = -l \), the interval becomes \( -l < x < l \) and the above results reduce to
\[ a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx \] and \[ b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \].
ILLUSTRATIVE EXAMPLES

Example 1. Find Fourier expansion for the function \( f(x) = x - x^2 \), \(-1 < x < 1\).

Sol. Let \( f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \)

Then
\[
\begin{align*}
a_0 &= \frac{1}{l} \int_{-1}^{1} (x - x^2) \, dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^{1} = \frac{2}{3} \\
a_n &= \frac{1}{l} \int_{-1}^{1} (x - x^2) \cos \frac{n\pi x}{l} \, dx = \left[ \frac{x^2}{2} \sin \frac{n\pi x}{l} - \frac{2x}{n\pi} \cos \frac{n\pi x}{l} + \left( \frac{2}{n^2\pi^2} - \frac{1}{n^2\pi^2} x^2 \right) \sin \frac{n\pi x}{l} \right]_{-1}^{1} \\
b_n &= \frac{1}{l} \int_{-1}^{1} (x - x^2) \sin \frac{n\pi x}{l} \, dx = \left[ \frac{x^2}{2} \cos \frac{n\pi x}{l} - \frac{2x}{n\pi} \sin \frac{n\pi x}{l} - \left( \frac{2}{n^2\pi^2} - \frac{1}{n^2\pi^2} x^2 \right) \cos \frac{n\pi x}{l} \right]_{-1}^{1} \\
\end{align*}
\]

Example 2. Find the Fourier series to represent \( f(x) = x^2 - 2 \), when \(-2 \leq x \leq 2\).

Sol. Since \( f(x) \) is an even function, \( b_n = 0 \).
Let \( f(x) = x^2 - 2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \)

Then
\[
\begin{align*}
a_0 &= \frac{1}{l}\int_{-1}^{1} (x^2 - 2) \, dx = \left[ \frac{x^3}{3} - 2x \right]_{-1}^{1} = \frac{4}{3} \\
a_n &= \frac{1}{l} \int_{-1}^{1} (x^2 - 2) \cos \frac{n\pi x}{l} \, dx = \left[ \frac{x^2}{n\pi} \sin \frac{n\pi x}{l} - \frac{2x}{n^2\pi^2} \cos \frac{n\pi x}{l} - \frac{2}{n^3\pi^3} \sin \frac{n\pi x}{l} \right]_{-1}^{1} \\
b_n &= \frac{1}{l} \int_{-1}^{1} (x^2 - 2) \sin \frac{n\pi x}{l} \, dx = \left[ \frac{x^2}{n\pi} \cos \frac{n\pi x}{l} + \frac{2x}{n^2\pi^2} \sin \frac{n\pi x}{l} - \frac{2}{n^3\pi^3} \cos \frac{n\pi x}{l} \right]_{-1}^{1} \\
\end{align*}
\]

Example 3. Expand \( f(x) = e^x \) as a Fourier series in the interval \((-1, 1)\).

Sol. Let \( f(x) = e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \)

Then
\[
\begin{align*}
a_0 &= \frac{1}{l} \int_{-1}^{1} e^x \, dx = \left[ \frac{e^x}{l} \right]_{-1}^{1} = \frac{1}{l} (e^1 - e^{-1}) = \frac{2 \sinh l}{l} \\
a_n &= \frac{1}{l} \int_{-1}^{1} e^x \cos \frac{n\pi x}{l} \, dx = \frac{1}{l} \left[ \frac{e^x}{n\pi} \left( -\cos \frac{n\pi x}{l} + \frac{n\pi x}{l} \sin \frac{n\pi x}{l} \right) \right]_{-1}^{1} \\
b_n &= \frac{1}{l} \int_{-1}^{1} e^x \sin \frac{n\pi x}{l} \, dx = \frac{1}{l} \left[ \frac{e^x}{n\pi} \left( -\sin \frac{n\pi x}{l} + \frac{n\pi x}{l} \cos \frac{n\pi x}{l} \right) \right]_{-1}^{1} \\
\end{align*}
\]

Example 4. For Fourier series of \( f(x) \),

\[ a_n = \frac{2}{l} \int_{0}^{l} (x^2 - 2) \cos \frac{n\pi x}{l} \, dx \]
\[ = \frac{2}{l} \left[ \frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} - 2x \frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} + 2 \frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l}} \right]_{0}^{l} \]
\[ = \frac{16 \cos \frac{n\pi}{l}}{n\pi} - \frac{16(-1)^n}{n^2\pi^2} \]
\[ e^x = \sinh \left( \frac{1}{l} - 2 \left( \frac{1}{2} \cos \frac{\pi x}{l} - \frac{3}{2} \cos \frac{2\pi x}{l} + \frac{1}{2} \cos \frac{3\pi x}{l} - \cdots \right) \right) \]

- \left( \frac{1}{l} + \frac{3}{l^2} \sin \frac{\pi x}{l} - \frac{3}{l^2} \sin \frac{2\pi x}{l} + \frac{3}{l^2} \sin \frac{3\pi x}{l} - \cdots \right) \]

Example 4. Obtain Fourier series for the function \( f(x) = \pi x \), \( 0 \leq x \leq l \),
\( f(x) = \pi (2-x) \), \( l \leq x \leq 2l \).


Sol. Let \( f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \)

Then
\[ a_0 = \int_0^l f(x) \, dx = \int_0^l \pi x \, dx + \int_l ^2 \pi (2-x) \, dx = \pi \left[ \frac{x^2}{2} \right]_0^l + \pi \left[ 2x - \frac{x^2}{2} \right]_l^2 \]

\[ a_0 = \pi \left( \frac{1}{2} \right) + \pi \left[ (4 - 2) - \left( - \frac{1}{2} \right) \right] = \pi \]

\[ a_n = \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx = \int_0^l \pi x \cos \frac{n\pi x}{l} \, dx + \int_l ^2 \pi (2-x) \cos \frac{n\pi x}{l} \, dx \]

\[ = \pi \left[ \frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} - n \left( - \frac{\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l + \pi \left[ \frac{2n\pi}{n\pi^2} \cos \frac{n\pi x}{n\pi} \right]_l^2 \]

\[ a_n = \pi \left[ \frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} - n \left( - \frac{\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l + \pi \left[ \frac{2n\pi}{n\pi^2} \cos \frac{n\pi x}{n\pi} \right]_l^2 \]

\[ = 0 \text{ or } -\frac{4}{n^2\pi^2} \text{ according as } n \text{ is even or odd.} \]

\[ b_n = \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx = \int_0^l \pi x \sin \frac{n\pi x}{l} \, dx + \int_l ^2 \pi (2-x) \sin \frac{n\pi x}{l} \, dx \]

\[ = \pi \left[ \frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} - n \left( -\frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l + \pi \left[ \frac{2n\pi}{n^n\pi^2} \cos \frac{n\pi x}{n^n\pi} \right]_l^2 \]

\[ b_n = \pi \left[ \frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} - n \left( -\frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l + \pi \left[ \frac{2n\pi}{n^n\pi^2} \cos \frac{n\pi x}{n^n\pi} \right]_l^2 \]

\[ f(x) = \frac{\pi}{2} + \frac{4}{\pi} \left( \frac{\cos \frac{\pi x}{l}}{1^2} + \frac{\cos \frac{3\pi x}{l}}{3^2} + \cdots \right) \]

Note. Putting \( x = 0 \), we have \( f(0) = \frac{\pi}{2} + \frac{4}{\pi} \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) \)

or
\[ \frac{\pi}{2} + \frac{4}{\pi} \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) = \frac{\pi^2}{8} \]

EXERCISE 1.3

1. Find a Fourier series for \( f(t) = 1 - t^2 \) when \( -1 < t < 1 \).

2. Expand \( f(x) \) in Fourier series in the interval \((-2, 2)\) when \( f(x) = 0 \), \(-2 < x < 0 \)
\[ = 1, \quad 0 < x < 2 \]

3. Develop \( f(x) \) in a Fourier series in the interval \((0, 2)\) if \( f(x) = x \), \( 0 < x < 1 \)
\[ = 0, \quad 1 < x < 2 \]

4. Find the Fourier expansion for \( f(x) = \pi x \) from \( x = -c \) to \( x = c \).

5. Find the Fourier expansion for the function \( f(x) = x - x^2 \) in the interval \(-1 < x < 1 \).

6. \( f(x) = \pi (2-x) \) for \( 0 < x < 1 \)

\[ = 1 - 4, \quad 1 < t < 2 \]

(b) Find the Fourier series for the function

\[ f(x) = \begin{cases} x, & -1 < x \leq 0 \\ x + 2, & 0 < x < 1 \end{cases} \]

(c) Find the Fourier series expansion of \( f(x) = 2x - x^2 \) in \((0, 3)\) and hence deduce that

\[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{12} \]

7. Find a Fourier series to represent \( x^2 \) in the interval \((-1, 1)\).

8. Expand \( f(x) = e^x \) as a Fourier series in the interval \((-2, 2)\).

9. Expand:

\[ f(x) = \begin{cases} \frac{1}{2} - x, & 0 < x < \frac{1}{2} \\ \frac{x}{4} - \frac{1}{2}, & \frac{1}{2} < x < 1 \end{cases} \]

as a Fourier series.

10. Find the Fourier series for the function \( f(x) = \begin{cases} k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases} \)

11. A sinusoidal voltage \( E \sin \omega t \) is passed through a half-wave rectifier which clips the negative portion of the wave.

Expand the resulting periodic function \( u(t) = \begin{cases} 0, & \frac{T}{2} < t < 0 \\ E \sin \omega t, & 0 < t < \frac{T}{2} \end{cases} \)

and \( T = \frac{2\pi}{\omega} \) in a Fourier series.

Answers

1. \( 1 - t^2 = \frac{2}{3} \pi^2 \left( \cos \frac{\pi x}{2^2} - \cos \frac{3\pi x}{2^2} + \cdots \right) \)
2. \( f(x) = \frac{1}{2} + \frac{2}{\pi} \left( \sin \frac{\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} + \frac{1}{5} \sin \frac{5\pi x}{2} + \ldots \right) \)

3. \( f(x) = \frac{1}{4} - \frac{2}{\pi^2} \left( \cos \frac{\pi x}{2} - \cos \frac{3\pi x}{2} + \ldots \right) + \frac{1}{\pi} \left( \sin \frac{\pi x}{2} + \sin \frac{2\pi x}{2} + \sin \frac{3\pi x}{2} + \ldots \right) \)

4. \( f(x) = 2x \left[ \sin \left( \frac{\pi x}{2} \right) - \frac{1}{3} \sin \left( \frac{3\pi x}{2} \right) + \frac{1}{5} \sin \left( \frac{5\pi x}{2} \right) - \ldots \right] \)

5. \( f(x) = \frac{12}{\pi^3} \left( \sin \frac{\pi x}{2} - \frac{2}{3} \sin \frac{2\pi x}{2} + \sin \frac{3\pi x}{2} + \ldots \right) \)

6. \( f(t) = \frac{4}{\pi^2} \left( \cos \frac{\pi t}{2} + \frac{3}{4} \cos \frac{3\pi t}{2} + \ldots \right) + \frac{1}{\pi} \left( \sin \frac{\pi t}{2} + \sin \frac{3\pi t}{2} + \ldots \right) \)

7. \( x^2 = \frac{1}{3} + \frac{4l^2}{\pi^2} \left( \cos \frac{\pi x}{2} - \frac{1}{3} \cos \frac{3\pi x}{2} + \frac{1}{5} \cos \frac{5\pi x}{2} - \ldots \right) \)

8. \( e^x = \sinh \frac{x}{2} - \frac{1}{4} \left( \frac{1}{2} \cos x - \frac{1}{2} \cos 2x + \frac{1}{2} \cos 3x - \frac{1}{2} \cos 4x + \ldots \right) - 2 \left( \frac{1}{2} \sin x - \frac{1}{2} \sin 2x + \frac{1}{2} \sin 3x - \frac{1}{2} \sin 4x + \ldots \right) \)

9. \( f(x) = \frac{7}{18} - \frac{1}{\pi} \left( \frac{1}{2} \cos x - \frac{1}{3} \cos 2x + \frac{1}{5} \cos 3x - \frac{1}{7} \cos 5x + \ldots \right) \)

10. \( f(x) = \frac{1}{2} + \frac{2k}{\pi} \left( \cos \frac{\pi x}{2} - \frac{1}{3} \cos \frac{3\pi x}{2} + \frac{1}{5} \cos \frac{5\pi x}{2} - \ldots \right) \)

11. \( u(t) = \frac{E}{\pi} \sin \omega t - \frac{2E}{\pi} \left( \cos 2\omega t + \cos 4\omega t + \cos 6\omega t + \ldots \right) \)

### 1.7. Half Range Series

Sometimes it is required to expand a function \( f(x) \) in the range \((0, \pi)\) in a Fourier series of period \(2\pi\) or more generally in the range \((0, L)\) in a Fourier series of period \(2L\).

If it is required to expand \( f(x) \) in the interval \((0, L)\), then it is immaterial what the function may be outside the range \(0 < x < L\). We are free to choose it arbitrarily in the interval \((-L, 0)\).

If we extend the function \( f(x) \) by reflecting it in the \(y\)-axis so that \( f(-x) = f(x) \), then the extended function is even for which \( b_n = 0 \). The Fourier expansion of \( f(x) \) will contain only cosine terms.

If we extend the function \( f(x) \) by reflecting it in the origin so that \( f(-x) = -f(x) \), then the extended function is odd for which \( a_0 = a_n = 0 \). The Fourier expansion of \( f(x) \) will contain only sine terms.

For example, consider the function

\[ f(x) = x, \quad 0 < x < \frac{\pi}{2} \]

\[ = \pi - x, \quad \frac{\pi}{2} < x < \pi \]

(Reflection in the \(y\)-axis)

(Reflection in the origin)

Hence a function \( f(x) \) defined over the interval \(0 < x < l\) is capable of two distinct half range series.

The half-range cosine series is \( f(x) = a_0 + \sum a_n \cos \frac{n\pi x}{l} \)

where \( a_0 = \frac{2}{l} \int_0^l f(x) \mathrm{d}x, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} \mathrm{d}x \)

The half-range sine series is \( f(x) = \sum b_n \sin \frac{n\pi x}{l} \), where \( b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \mathrm{d}x \)

Cor. If the range is \(0 < x < \pi\), then

(i) The half-range cosine series is \( f(x) = a_0 + \sum a_n \cos nx \)

where \( a_0 = \frac{2}{\pi} \int_0^\pi f(x) \mathrm{d}x, \quad a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \mathrm{d}x \).

(ii) The half-range sine series is \( f(x) = \sum b_n \sin nx \), where \( b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \mathrm{d}x \).

(iii)
Example 1. Expand \( nx - x^2 \) in a half-range sine series in the interval \((0, \pi)\) up to the first three terms.

Sol. Let \( nx - x^2 = \sum_{n=1}^{\infty} b_n \sin nx \), then

\[
b_n = \frac{2}{\pi} \int_0^\pi (nx - x^2) \sin nx \, dx
\]

\[
= \frac{2}{\pi} \left[ \left( \frac{nx^2}{n} - \frac{nx^3}{3} \right) - \left( \frac{x^3}{3} - \frac{x^5}{5} \right) \right]_0^\pi
\]

\[
= \frac{2}{\pi} \left[ \frac{1}{n} - \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right] - \frac{2}{\pi} \left( \frac{\cos nx}{n^3} \right)_0^\pi
\]

\[
= \frac{2}{\pi} \left[ \frac{1}{n} - \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right] - \frac{2}{\pi} \left( \frac{\cos nx}{n^3} \right)
\]

\[
= 0 \quad \text{or} \quad \frac{8}{\pi n^3} \quad \text{according as } n \text{ is even or odd}
\]

\[
\therefore \quad nx - x^2 = \frac{8}{\pi} \left( \sin x + \sin 3x + \sin 5x + \cdots \right).
\]

Example 2. If \( f(x) = x \), \( 0 < x < \frac{\pi}{2} \),

\[
\pi - x = \frac{\pi}{2} < x < \pi
\]

show that (i) \( f(x) = \frac{8}{\pi} \left[ \sin x - \sin 3x + \sin 5x + \cdots \right] \)

(ii) \( f(x) = \frac{2}{\pi} \left[ \cos 2x + \cos 6x + \cos 10x + \cdots \right] \).

Sol. (i) For the half-range sine series

Let \( f(x) = \sum_{n=1}^{\infty} b_n \sin nx \)

Then \( b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^\frac{x}{2} x \sin nx \, dx + \int_{\frac{x}{2}}^\pi (\pi - x) \sin nx \, dx \)

\[
= \frac{2}{\pi} \int_0^\frac{x}{2} x \left( \frac{\cos nx}{n} - \frac{\sin nx}{n^2} \right) \, dx + \frac{2}{\pi} \left[ \left( \frac{\cos nx}{n} - \frac{\sin nx}{n^2} \right) \right]_{\frac{x}{2}}^\pi
\]

\[
= \frac{2}{\pi} \left[ \frac{x}{2} \cos nx + \frac{\sin nx}{n^2} \right]_{\frac{x}{2}}^\pi
\]

When \( n \) is even, \( b_n = 0 \).

\[
\therefore \quad f(x) = \frac{4}{\pi} \left[ \sin x - \sin 3x + \sin 5x + \cdots \right].
\]

(ii) For the half-range cosine series

Let \( f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \)

Then \( a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^\frac{x}{2} x \cos nx \, dx + \int_{\frac{x}{2}}^\pi (\pi - x) \cos nx \, dx \)

\[
= \frac{2}{\pi} \left[ \left( \frac{x}{2} \cos nx + \frac{\sin nx}{n^2} \right) \right]_{\frac{x}{2}}^\pi
\]

\[
= \frac{2}{\pi} \left[ \frac{n^2}{8} + \frac{n^2}{8} \right] - \frac{2}{\pi} \left( \frac{n^2}{8} \right)
\]

\[
a_n = \frac{2}{\pi} \left[ \sin nx \right]_{\frac{x}{2}}^\pi - \frac{2}{\pi} \left( \frac{n^2}{8} \right)
\]

\[
\therefore \quad a_1 = 0, a_2 = \frac{2}{\pi} \left( 2 \cos \pi \cos 2\pi - 1 \right) = -\frac{2}{\pi}.
\]

Example 3. Find a series of cosines of multiples of \( x \) which will represent \( x \sin x \) in the interval \((0, \pi)\) and show that \( \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots = \frac{\pi - 2}{4} \).

Sol. Let \( x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \)

Then \( a_0 = \frac{2}{\pi} \int_0^\pi x \sin x \, dx = \frac{2}{\pi} \left[ x \cos x - \frac{x^2}{2} \right]_0^\pi = \frac{2}{\pi} \left[ -\pi \cos x \right] = 2 \)

\[
a_n = \frac{2}{\pi} \int_0^\pi x \sin nx \cos nx \, dx = \frac{1}{\pi} \int_0^\pi x (2 \cos nx \sin x) \, dx
\]
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$$= \frac{2}{1 + n^2 \pi^2} \left[ -\sin \pi (-1)^n + n \pi \right] = \frac{2n \pi}{1 + n^2 \pi^2} \left[ 1 - e (-1)^n \right]$$

Hence $$e^n = 2 \sum_{n=1}^{\infty} \frac{n \sin \pi x}{1 + n^2 \pi^2}$$

$$= 2 \pi \left[ \frac{1 + e}{1 + \pi^2} \sin \pi x + \frac{2(1 - e)}{1 + 4 \pi^2} \sin 2\pi x + \frac{3(1 + e)}{1 + 9 \pi^2} \sin 3\pi x + \ldots \right]$$

Example 5. Develop $$\sin \left( \frac{\pi x}{l} \right)$$ in half-range cosine series in the range $$0 < x < l$$

Sol. Let $$\sin \left( \frac{\pi x}{l} \right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n \pi x}{l}$$

then $$a_0 = \frac{2}{l} \int_0^l \sin \left( \frac{\pi x}{l} \right) \cos \left( \frac{\pi x}{l} \right) dx = \frac{2}{l} \left[ -\frac{\cos \frac{\pi x}{l}}{\pi} \right]_0^l = -\frac{2}{l} \left[ \cos \pi - 1 \right] = -\frac{4}{\pi}$$

$$a_n = \frac{2}{l} \int_0^l \sin \left( \frac{\pi x}{l} \right) \cos \left( \frac{n \pi x}{l} \right) \cos \left( \frac{\pi x}{l} \right) dx$$

$$= \frac{1}{l} \left[ \cos \left( \frac{n+1}l \pi \right) \sin \left( \frac{n-1}l \pi \right) \right]_0^l$$

$$= \frac{1}{l} \left[ \frac{\cos \left( n+1 \right) \pi l}{n+1} + \frac{\cos \left( n-1 \right) \pi l}{n-1} \right]_{-1}^1$$

When $$n = 1$$, we have

$$a_1 = \frac{2}{l} \int_0^l \sin \frac{\pi x}{l} \cos \frac{\pi x}{l} dx = \frac{1}{l} \left[ \frac{\cos \frac{2\pi x}{l}}{\pi} \right]_0^l = -\frac{2}{\pi} \left[ \cos 2\pi - 1 \right] = -\frac{4}{\pi}$$
When $n$ is odd, $n \neq 1$, $a_n = \frac{1}{\pi} \left[ \frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} \right] = 0$, also $a_1 = 0$

When $n$ is even, $a_n = \frac{1}{\pi} \left[ \frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} \right] = \frac{-4}{\pi(n+1)(n-1)}$

$= \frac{4}{\pi(n+1)(n-1)}$

$\therefore \sin \left( \frac{\pi x}{l} \right) = \frac{2}{\pi} \left[ \frac{\cos \frac{2\pi x}{l}}{1.3} + \frac{\cos \frac{4\pi x}{l}}{3.5} + \frac{\cos \frac{6\pi x}{l}}{5.7} + \ldots \right].$

Example 6. Obtain a half-range cosine series for

$f(x) = kx \quad \text{for} \quad 0 \leq x \leq \frac{l}{2}$

$= k(l-x) \quad \text{for} \quad \frac{l}{2} \leq x \leq l.$


Deduce the sum of the series $\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{5^2} + \ldots$ (Dec. 2005, May 2008, Dec. 2006)

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

then $a_0 = \frac{2}{l} \int_0^{l/2} f(x) dx = \frac{2}{l} \int_0^{l/2} kx dx + \frac{1}{l} k(l-x) dx$

$= \frac{2}{l} \left[ \frac{kx^2}{2} \right]_0^{l/2} + k \left[ \frac{lx - x^2}{2} \right]^{l/2}$

$= \frac{2}{l} \frac{kl^2}{8} + k \left[ \frac{l^2}{2} - \frac{l^2}{2} \right] - k \left[ \frac{l^2}{8} - \frac{l^2}{8} \right] = \frac{2}{l} \left( \frac{kl^2}{4} \right) = \frac{kl^2}{2}$

$a_n = \frac{2}{l} \int_0^{l/2} f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{l} \int_0^{l/2} kx \cos \frac{n\pi x}{l} dx + \int_0^{l/2} k(l-x) \cos \frac{n\pi x}{l} dx$

$= \frac{2}{l} \left[ kx \frac{1}{n\pi} \sin \frac{n\pi x}{l} + k \frac{l^2}{n^2 \pi^2} \cos \frac{n\pi x}{l} \right]_0^{l/2}$

$\therefore f(x) = \frac{kl^2}{2} \sin \frac{n\pi}{l} + \frac{kl^2}{n^2 \pi^2} \left( \cos \frac{n\pi}{l} - 1 \right)$

When $n$ is odd, $\cos \frac{2n\pi}{l} = 0$ and $\cos \frac{n\pi}{l} = -1 \Rightarrow a_n = 0 \Rightarrow a_1 = a_3 = a_5 = \ldots = 0$

When $n$ is even, $a_n = \frac{2kl^2}{n^2 \pi^2} \left[ \cos \frac{2n\pi}{l} - 1 \right] = \frac{-2kl^2}{n^2 \pi^2}$

Putting $x = l, f(l) = 0$

$\therefore$ From (1), we have $0 = \frac{kl^2}{4} \frac{8kl}{\pi^2} \left( \frac{1}{2^2} + \frac{1}{6^2} + \ldots \right)$

$\Rightarrow \frac{1}{2^2} + \frac{1}{6^2} + \ldots = \frac{\pi^2}{8}$

Hence $\frac{1}{2^2} + \frac{1}{6^2} + \ldots = \frac{\pi^2}{8}$

EXERCISE 1.4

1. (a) Obtain cosine and sine series for $f(x) = x$ in the interval $0 \leq x \leq \pi$. Hence show that

$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \frac{1}{n^2} = \frac{\pi^2}{8}$

(b) Prove that for $0 < x < l$

$x = \frac{l}{2} - \frac{4l}{\pi^2} \left( \cos \frac{\pi x}{l} + \frac{3\pi x}{l} \cos \frac{5\pi x}{l} + \ldots \right)$

2. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \leq x \leq \pi$.

3. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$.

Hence show that

\begin{align*}
(i) \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots &= \frac{\pi^2}{6} \\
(ii) \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots &= \frac{\pi^2}{12} \\
(iii) \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots &= \frac{\pi^2}{8}
\end{align*}

(M.D.U. 2006)

4. (a) Express \( \sin x \) as a cosine series in \( 0 < x < \pi \).

(b) Show that a constant function \( c \) can be expanded in an infinite series

\[ \frac{4c}{\pi} \left( \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right) \]  

in the range \( 0 < x < \pi \).

(c) Find the Fourier half-range sine series of \( f(x) = 1 \), \( 0 \leq x \leq 2 \).  

(M.D.U. May 2008)

5. If 

\[ f(x) = \begin{cases} 
\frac{\pi}{3}, & 0 \leq x \leq \frac{\pi}{3} \\
0, & \frac{\pi}{3} < x < \frac{2\pi}{3} \\
-\frac{\pi}{3}, & \frac{2\pi}{3} \leq x \leq \pi 
\end{cases} \]

then show that 

\[ f(x) = \frac{2}{\sqrt{3}} \left[ \cos x - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} - \cdots \right]. \]

6. If 

\[ f(x) = mx, \quad 0 \leq x \leq \frac{\pi}{2} \]

\[ = m(\pi - x), \quad \frac{\pi}{2} \leq x \leq \pi \]

then show that 

\[ f(x) = \frac{4m}{\pi} \left[ \sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots \right]. \]

7. Express \( f(x) = x \) as a half-range.

(i) sine series in \( 0 < x < 2 \).  

(ii) cosine series in \( 0 < x < 2 \).  


8. Find the Fourier sine and cosine series of

\[ f(x) = \begin{cases} 
x, & 0 < x < \frac{\pi}{2} \\
0, & \frac{\pi}{2} < x < \pi 
\end{cases} \]

(M.D.U. Dec. 2008)

9. Show that the series 

\[ \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2\pi nx}{l} \]  

represents \( \frac{1}{2}l - x \) when \( 0 < x < l \).

10. Find the half-range sine series for 

\[ f(x) = \frac{1}{4} - x, \quad 0 < x < \frac{1}{2} \]

\[ = x - \frac{3}{4}, \quad \frac{1}{2} < x < 1. \]

(V.T.U. 2006)

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11. Represent the following function by Fourier sine series

\[ f(x) = \begin{cases} 
1, & 0 < x < \frac{l}{2} \\
0, & \frac{l}{2} < x < l
\end{cases} \]

12. Find the half-range sine series for the function \( f(t) = t - t^2 \), \( 0 < t < 1 \).

13. Prove that for \( 0 < x < \pi \),

\[ x(\pi - x) = \frac{\pi^2}{6} \left( \cos \frac{2x}{\pi} + \cos \frac{4x}{\pi} + \cos \frac{6x}{\pi} + \cdots \right) \]

14. Let \( f(x) = \begin{cases} 
x, & 0 \leq x \leq \frac{l}{2} \\
0, & \frac{l}{2} < x \leq l
\end{cases} \)

Show that 

\[ f(x) = \frac{4l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)x}{l} \]

Hence obtain the sum of the series

\[ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \]

15. If \( f(x) = \begin{cases} 
\sin x, & 0 \leq x < \frac{\pi}{4} \\
\cos x, & \frac{\pi}{4} \leq x < \frac{\pi}{2}
\end{cases} \)

expand \( f(x) \) in a series of sines.

16. For the function defined by the graph OAB, find the half-range Fourier sine series.

\[ \text{Answers} \]

1. \[ \begin{align*}
\left( \frac{\pi}{2} \right) & \frac{4}{\pi} \left( \cos x + \cos 3x + \cos 5x + \cdots \right) \\
2 & \left( \sin x - \sin 2x + \sin 3x - \sin 4x + \cdots \right)
\end{align*} \]

2. \[ \frac{\pi^2}{3} - 4 \left[ \cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots \right] \]
1.8. PARSEVAL'S THEOREM ON FOURIER CONSTANTS

If the Fourier series of \( f(x) \) over an interval \( c < x < c + 2l \) is given as

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n \pi x}{l} + b_n \sin \frac{n \pi x}{l} \right]
\]

then

\[
\frac{1}{2l} \int_{c}^{c+2l} [f(x)]^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).
\]

Proof. The Fourier series of \( f(x) \) in \( c < x < c + 2l \) is given as

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n \pi x}{l} + b_n \sin \frac{n \pi x}{l} \right]
\]

where

\[
a_0 = \frac{1}{l} \int_{c}^{c+2l} f(x) \, dx; \quad a_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \cos \frac{n \pi x}{l} \, dx; \quad b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin \frac{n \pi x}{l} \, dx.
\]

Multiplying both sides of (1) by \( f(x) \), we have

\[
[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos \frac{n \pi x}{l} + \sum_{n=1}^{\infty} b_n f(x) \sin \frac{n \pi x}{l}
\]
ILLUSTRATIVE EXAMPLES

Example 1. Find the Fourier sine series for unity in \(0 < x < \pi\) and hence show that
\[
1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.
\]

Sol. We require half-range Fourier sine series for 1 in \((0, \pi)\)

Let
\[
1 = \sum_{n=1}^{\infty} b_n \sin nx
\]

Then
\[
b_n = \frac{2}{\pi} \int_{0}^{\pi} (1) \sin nx \, dx = \frac{2}{\pi} \left[ -\cos nx \right]_{0}^{\pi} \frac{\cos n\pi - 1}{n}
\]

\[
= \frac{2}{\pi n} (1 - (-1)^n)
\]

Now \(b_n = 0\) when \(n\) is even; and \(b_n = \frac{4}{n\pi}\) when \(n\) is odd.

Substituting in (1), we get

\[
1 = \sum_{n=1}^{\infty} \frac{4}{(2m-1)\pi} \sin (2m-1)x \quad \text{or} \quad 1 = \frac{4}{\pi} \left( \sin x + \sin 3x + \sin 5x + \ldots \right)
\]

Now from Parseval's theorem on Fourier constants

\[
\int_{0}^{\pi} [f(x)]^2 \, dx = \pi \left[ \frac{a_0^2}{1} + \frac{a_2^2}{2} + \frac{b_1^2}{1} + \frac{a_4^2}{2} + \frac{b_3^2}{1} + \ldots \right]
\]

Applying (3) to half-range sine series for 1 in \((0, \pi)\)

\[
c = 0, 2l = \pi, f(x) = 1, a_0 = 0, a_n = 0, \text{ and } b_n = \frac{4}{(2m-1)\pi}, m = 1, 2, \ldots
\]

We get,

\[
\int_{0}^{\pi} 1^2 \, dx = \pi \left[ \frac{1}{2} \sum_{m=1}^{\infty} \frac{16}{(2m-1)^2} \right] \left[ \pi^2 \right] = \frac{\pi^4}{90} = \left( \frac{1}{2^4} + \frac{1}{4^4} + \ldots \right)
\]

Hence the result.

Example 2. Find Fourier series of \(x^2\) in \((-\pi, \pi)\). Use Parseval's identity to prove that

\[
\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots
\]

Sol. The Fourier series of \(x^2\) in \((-\pi, \pi)\) is

\[
x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx
\]

Here

\[
a_0 = \frac{2\pi^2}{3}, a_n = \frac{4(-1)^n}{n^2}, b_n = 0, f(x) = x^2
\]

Now by Parseval's identity from (1), we get

\[
\int_{-\pi}^{\pi} [x^2]^2 \, dx = 2x \left[ \frac{\pi^4}{9} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{n^4} \right]
\]

\[
\Rightarrow \left[ \frac{x^5}{5} \right]_{-\pi}^{\pi} = \frac{2\pi^5}{9} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{n^4} \quad \text{or} \quad \frac{2\pi^5}{9} - \frac{2\pi^5}{9} = \pi \sum_{n=1}^{\infty} \frac{16}{n^4}
\]

or

\[
\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \pi^4
\]

EXERCISE 1.5

1. If \(f(x)\) has the Fourier series expansion

\[
a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \quad \text{in} \quad a \leq x \leq a + 2l
\]

show that

\[
\int_{a}^{a+2l} [f(x)]^2 \, dx = 2l \left[ \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right) \right]
\]

2. If \(f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}\) in \(0 < x < l\), then show that

\[
\int_{0}^{l} [f(x)]^2 \, dx = \frac{l}{2} \left[ \frac{a_0^2}{4} + \sum_{n=1}^{\infty} a_n^2 \right]
\]

3. If \(f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\) in \((0, l)\), then show that

\[
\int_{0}^{l} [f(x)]^2 \, dx = \frac{l}{2} \sum_{n=1}^{\infty} b_n^2
\]

4. Prove that in the range \((0, l)\),

\[
x = \frac{l}{2} \left[ \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2m-1)^2} \cos \frac{(2m-1)\pi x}{l} \right]
\]

and deduce that

\[
\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}
\]
5. Show that for $0 < x < \pi$,  
   
   \[
   x(x - \pi) = \frac{\pi^2}{6} \left( \frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots \right)
   \]

   And hence evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

### 1.9. Typical Waveforms

A periodic waveform is a waveform that repeats a basic pattern. It is a single-valued periodic function. Therefore, it can be developed as a Fourier series.

We give below some typical waveforms usually met with in communication engineering and also the corresponding Fourier series. The student is urged to construct the Fourier series in each case.

#### I. Square (or Rectangular) Waveform

It is a periodic function of the form given below.

**(i)**

\[
\begin{align*}
  f(x) &= \begin{cases} 
  -k & \text{for } -\pi < x < 0 \\
  k & \text{for } 0 < x < \pi 
  \end{cases} \\
  f(x + 2\pi) &= f(x)
\end{align*}
\]

![Graph of the square waveform](image)

Its Fourier expansion is

\[
  f(x) = \frac{4k}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right)
\]

[See Question 1 in Exercise 1.2]

**(ii)**

\[
  f(x) = \begin{cases} 
  1 & \text{when } 0 < x < \pi \\
  0 & \text{when } \pi < x < 2\pi 
  \end{cases} \\
  f(x + 2\pi) &= f(x)
\]

![Graph of the square waveform](image)

#### II. Saw-toothed Waveform

It is a periodic function of the form given below.

**(i)**

\[
  f(x) = x, \quad -\pi < x < \pi \quad \text{and} \quad f(x + 2\pi) = f(x)
\]

Its Fourier expansion is

\[
  f(x) = 2 \left( \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \ldots \right)
\]

#### Fourier Series

- **Its Fourier expansion is**

  \[
  f(x) = \frac{1}{2} \left[ \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right]
  \]

- **For** $-\pi < x < 0$

  \[
  f(x) = \frac{1}{2} \left[ \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right]
  \]

- **For** $0 < x < \pi$

  \[
  f(x) = \frac{1}{2} \left[ \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right]
  \]

[See Question 2 in Exercise 1.2]
II. Modified Saw-toothed Waveform

It is a periodic function of the form given below.

\[ f(x) = \begin{cases} \pi + x & \text{for } -\pi < x < 0 \\ 0 & \text{for } 0 \leq x < \pi \end{cases}, \quad f(x + 2\pi) = f(x) \]

Its Fourier expansion is

\[ f(x) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n} \text{, where } \omega = \frac{2\pi}{T}. \]

III. Triangular Waveform

It is a periodic function of the form given below.

\[ f(x) = \begin{cases} \frac{2 + x}{2} & \text{for } -2 \leq x \leq 0 \\ \frac{2 - x}{2} & \text{for } 0 < x \leq 2 \end{cases}, \quad f(x + 4) = f(x) \]

Its Fourier expansion is

\[ f(x) = 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \left(\frac{(2n - 1)\pi x}{2}\right) \]

\[ f(x) = \begin{cases} 1 + \frac{2x}{\pi} & \text{for } -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi} & \text{for } 0 < x \leq \pi \end{cases}, \quad f(x + 2\pi) = f(x) \]

Its Fourier expansion is

\[ f(x) = \frac{8}{\pi^2} \left( \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots \right) \]

[See Example 10 before Exercise 1.1]
V. Half Rectified Waveform

It is a periodic function of the form given below.

\[ f(x) = \begin{cases} 
0 & \text{for } -\pi \leq x < 0 \\
\sin x & \text{for } 0 \leq x \leq \pi 
\end{cases} \]

(i)

Its Fourier expansion is

\[ f(x) = \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1} \]

(ii)

\[ f(x) = \begin{cases} 
L \sin x & \text{for } 0 \leq x \leq \pi \\
0 & \text{for } \pi \leq x \leq 2\pi 
\end{cases} \, , \, f(x + 2\pi) = f(x) \]

Its Fourier expansion is

\[ f(x) = \frac{L}{\pi} + \frac{1}{2} L \sin x - \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1} \]

[See Question 6 in Exercise 1.2]

VI. Full Rectified Waveform

It is a periodic function of the form given below.

\[ f(x) = a \sin x \text{ for } 0 \leq x \leq \pi, \, f(x + \pi) = f(x) \]

\[ f(x) = \frac{4a}{\pi} \left[ \frac{1}{2} \cos 2x + \frac{\cos 4x}{1.3} + \frac{\cos 6x}{3.5} + \frac{5.7}{5} + \ldots \right] \]
2.1. INTEGRAL TRANSFORMS

The integral transform of a function \( f(x) \) is defined by the equation

\[
L[f(x)] = \hat{f}(s) = \int_0^\infty f(x) K(s, x) \, dx,
\]

where \( K(s, x) \) is a known function of \( s \) and \( x \), called the kernel of the transform; \( s \) is called the parameter of the transform and \( f(x) \) is called the inverse transform of \( \hat{f}(s) \).

Some of the well-known transforms are given below:

(i) **Laplace Transform.** When \( K(s, x) = e^{-sx} \), we have the Laplace transform of \( f(x) \).

Thus

\[
L[f(x)] = \hat{f}(s) = \int_0^\infty f(x) e^{-sx} \, dx
\]

(ii) **Fourier Transform.** When \( K(s, x) = e^{jx} \), we have the Fourier transform of \( f(x) \).

Thus

\[
F(s) = \int_0^\infty f(x) e^{jx} \, dx
\]

(iii) **Hankel Transform.** When \( K(s, x) = x J_n(sx) \), we have the Hankel transform of \( f(x) \).

Thus

\[
H_n(s) = \int_0^\infty f(x) x J_n(sx) \, dx
\]

where \( J_n(sx) \) is the Bessel function of the first kind and order \( n \).

(ii) **Mellin Transform.** When \( K(s, x) = x^{-1} \), we have the Mellin transform of \( f(x) \).

Thus

\[
M(s) = \int_0^\infty f(x) x^{-s} \, dx
\]

(c) **Fourier Sine Transform.** When \( K(s, x) = \sin sx \), we have the Fourier sine transform of \( f(x) \).

Thus

\[
F_s(s) = \int_0^\infty f(x) \sin sx \, dx
\]

(d) **Fourier Cosine Transform.** When \( K(s, x) = \cos sx \), we have the Fourier cosine transform of \( f(x) \).

Thus

\[
F_c(s) = \int_0^\infty f(x) \cos sx \, dx
\]

We have already discussed Laplace transform and its applications to the solution of ordinary differential equations. In the present chapter, we shall discuss the Fourier integrals and Fourier transforms which are useful in solving boundary value problems arising in engineering, e.g., conduction of heat, theory of communication, wave propagation, etc. Fourier the function is non-periodic. A suitable representation for non-periodic functions can be obtained by considering the limiting form of Fourier series when the fundamental period is made infinite. In such case, the Fourier series becomes a Fourier integral which can be expressed in terms of Fourier transform, which transforms a non-periodic function.

The effect of applying an integral transform to a partial differential equation is to reduce the number of independent variables by one. The choice of a particular transform is decided by the nature of the boundary conditions and the facility with which the transform can be inverted to give \( f(x) \).

2.2. FOURIER INTEGRAL THEOREM

**Statement.**

(i) \( f(x) \) satisfies Dirichlet's conditions in every interval \((-\epsilon, \epsilon)\) however large.

(ii) \[
\int_{-\infty}^{\infty} |f(x)| \, dx \text{ converges;}
\]

then

\[
f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) \, dt \, dx.
\]

The integral on the right hand side is called Fourier integral of \( f(x) \).

**Proof.** Consider a function \( f(x) \) satisfying Dirichlet's conditions in every interval \((-\epsilon, \epsilon)\), however large. Then

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right) \quad (1)
\]

where \( a_0 = \frac{1}{c} \int_{-\epsilon}^{\epsilon} f(t) \, dt, \quad a_n = \frac{1}{c} \int_{-\epsilon}^{\epsilon} f(t) \cos \frac{n\pi t}{c} \, dt \quad \text{ and } \quad b_n = \frac{1}{c} \int_{-\epsilon}^{\epsilon} f(t) \sin \frac{n\pi t}{c} \, dt \quad (2)
\]

Substituting the values of \( a_0, a_n \) and \( b_n \) in (1), we get

\[
f(x) = \frac{1}{2c} \int_{-\epsilon}^{\epsilon} f(t) \, dt + \frac{1}{c} \sum_{n=1}^{\infty} \int_{-\epsilon}^{\epsilon} \left[ \cos \frac{n\pi t}{c} \cos \frac{n\pi x}{c} + \sin \frac{n\pi t}{c} \sin \frac{n\pi x}{c} \right] f(t) \, dt \quad (3)
\]

If we assume that the integral \( \int_{-\epsilon}^{\epsilon} |f(t)| \, dt \), \( dx \) converges, then

\[
\lim_{\epsilon \to 0} \left[ \frac{1}{2c} \int_{-\epsilon}^{\epsilon} f(t) \, dt \right] = 0,
\]

since \[
\frac{1}{2c} \int_{-\epsilon}^{\epsilon} f(t) \, dt \leq \frac{1}{2c} \int_{-\epsilon}^{\epsilon} |f(t)| \, dt.
\]

Putting \( \frac{\pi}{c} = \Delta \lambda \), the second term in (2) becomes

\[
\frac{1}{\pi} \sum_{n=1}^{\infty} \Delta \lambda \int_{-\epsilon}^{\epsilon} \cos (n\pi \lambda (t-x)) f(t) \, dt
\]

This is of the form \[
\sum_{n=1}^{\infty} F(n\pi \lambda) \Delta \lambda \text{ whose limit as } \Delta \lambda \to 0, \text{ is } \int_{-\epsilon}^{\epsilon} F(\lambda) \, d\lambda.
\]
Hence as \( c \to \infty \), (2) reduces to

\[
\int_{-\infty}^{\infty} f(t) \cos \lambda(t - x) \, dt \, dx.
\]

which is known as Fourier Integral of \( f(x) \).

Equation (3) is true at a point of continuity. At a point of discontinuity the value of the integral on the right is

\[
\frac{1}{2} [f(x_1 + 0) + f(x_1 - 0)].
\]

### 2.3. FOURIER SINE AND COSINE INTEGRALS

We know that \( \cos \lambda(t - x) = \cos \lambda t \cos \lambda x + \sin \lambda t \sin \lambda x \)

\[
\therefore \text{ Fourier integral of } f(x) \text{ can be written as}
\]

\[
f(x) = \frac{1}{\pi} \int_{0}^{\infty} f(t) \cos \lambda t \cos \lambda x \, dt \, d\lambda.
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} \cos \lambda x \int_{0}^{\infty} f(t) \cos \lambda t \, dt \, d\lambda + \frac{1}{\pi} \int_{0}^{\infty} \sin \lambda x \int_{0}^{\infty} f(t) \sin \lambda t \, dt \, d\lambda.
\]

(4)

When \( f(x) \) is an odd function, \( f(t) \sin \lambda t \) is odd while \( f(t) \cos \lambda t \) is even. Thus the first integral in (4) vanishes and, we get

\[
f(x) = \frac{2}{\pi} \int_{0}^{\infty} \sin \lambda x \int_{0}^{\infty} f(t) \sin \lambda t \, dt \, d\lambda.
\]

(5)

This is called Fourier sine integral.

When \( f(x) \) is an even function, \( f(t) \cos \lambda t \) is even while \( f(t) \sin \lambda t \) is odd. Thus the second integral in (4) vanishes and, we get

\[
f(x) = \frac{2}{\pi} \int_{0}^{\infty} \cos \lambda x \int_{0}^{\infty} f(t) \cos \lambda t \, dt \, d\lambda.
\]

(6)

This is called Fourier cosine integral.

### 2.4. COMPLEX FORM OF FOURIER INTEGRAL

Since \( \cos \lambda(t - x) \) is an even function of \( \lambda \), we have from (3)

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \cos \lambda(t - x) \, dt \, d\lambda.
\]

(7)

Also \( \sin \lambda(t - x) \) is an odd function of \( \lambda \), so that

\[
0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \sin \lambda(t - x) \, dt \, d\lambda.
\]

(8)

Multiplying (8) by \( i \) and adding to (7), we get

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\lambda t} \, dt \, d\lambda.
\]

(9)

which is known as the complex form of Fourier integral.

### ILLUSTRATIVE EXAMPLES

Example 1. Express the function \( f(x) = \begin{cases} 1 & \text{for } |x| \leq 1, \\ 0 & \text{for } |x| > 1 \end{cases} \)

as a Fourier integral. Hence evaluate \( \int_{0}^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} \, d\lambda \).

(M.D.U May 2008)

Sol. The Fourier integral for \( f(x) \) is

\[
\frac{1}{\pi} \int_{0}^{\infty} f(t) \cos \lambda(t - x) \, dt \, d\lambda.
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\pi} \int_{0}^{\infty} f(t) \cos \lambda(t - x) \, dt \, d\lambda.
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\pi} \int_{0}^{\infty} f(t) \frac{\cos \lambda(t - x)}{\lambda} \, dt \, d\lambda.
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} f(t) \frac{\cos \lambda(t - x)}{\lambda} \, dt \, d\lambda.
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\pi} \int_{0}^{\infty} f(t) e^{i\lambda t - x} \, dt \, d\lambda.
\]

(9)

Example 2. Express \( f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \pi, \\ 0 & \text{for } x > \pi \}

as a Fourier sine integral and hence evaluate

\[
\int_{0}^{\pi} \frac{1 - \cos (n\lambda)}{\lambda} \sin (n\lambda) \, d\lambda.
\]

(Kottayam 2005)
Sol. The Fourier sine integral for $f(x)$ is
\[
\frac{2}{\pi} \int_{0}^{\pi} \sin(\lambda x) \left[ \int_{0}^{\pi} f(t) \sin(\lambda t) \, dt \right] \, d\lambda
\]
\[
= \frac{2}{\pi} \int_{0}^{\pi} \sin(\lambda x) \left[ \int_{0}^{\pi} f(t) \sin(\lambda t) \, dt \right] \, d\lambda
\]
\[
= \frac{2}{\pi} \int_{0}^{\pi} \sin(\lambda x) \left[ \frac{-\cos(\lambda t)}{\lambda} \right]_{0}^{\pi} \, d\lambda
\]
\[
= \frac{2}{\pi} \int_{0}^{\pi} \frac{1 - \cos(\lambda t)}{\lambda} \sin(\lambda x) \, d\lambda
\]
\[
f(x) = \frac{2}{\pi} \int_{0}^{\pi} \frac{1 - \cos(\lambda x)}{\lambda} \sin(\lambda x) \, d\lambda
\]
\[
= \frac{1}{\pi} \int_{0}^{\pi} \cos(\alpha x) \sin(\alpha x) \, d\alpha = \frac{1}{\pi} \int_{0}^{\pi} \cos(\alpha x) \sin(\alpha x) \, d\alpha
\]
(Replacing $\lambda$ by $\alpha$)

When $x = 0$,
\[
\int_{0}^{\pi} \cos(\alpha x) \sin(\alpha x) \, d\alpha = \frac{1}{1 + \alpha^2} \int_{0}^{\pi} \cos(\alpha x) \sin(\alpha x) \, d\alpha
\]
\[
= \frac{1}{1 + \alpha^2} \left[ \tan^{-1} \alpha \right]_{0}^{\pi} = \frac{\pi}{2}
\]

EXERCISE 2.1

Using Fourier integral representation, show that (1 - 9):

1. $\int_{0}^{\pi} \frac{\lambda \sin x \lambda}{\lambda^2 + x^2} \, d\lambda = \frac{\pi}{2} e^{-\lambda x}, x > 0, \lambda > 0$
2. $\int_{0}^{\pi} \frac{\sin \lambda x}{1 + \alpha^2} \, d\lambda = \frac{\pi}{2} e^{-\alpha x}, x > 0$
3. $\int_{0}^{\pi} \frac{\sin \lambda x}{1 - \lambda^2} \, d\lambda = \frac{\pi}{2} \sin x, \text{ when } 0 \leq x \leq \pi$
4. $\int_{0}^{\pi} \frac{\cos \lambda x}{\lambda^2 + x^2} \, d\lambda = \frac{\pi}{2} e^{-\lambda x}, \lambda > 0, x > 0$
5. $\int_{0}^{\pi} \frac{\lambda x^2 + \lambda^2}{\lambda^2 + 4} \cos \lambda x \, d\lambda = \frac{\pi}{2} e^{-\lambda x} \cos x, x > 0$
6. $\int_{0}^{\pi} \frac{\lambda^3}{\lambda^2 + 4} \sin \lambda x \, d\lambda = \frac{\pi}{2} e^{-\lambda x} \sin x, x > 0$
7. $\int_{0}^{\pi} \frac{\lambda \sin \lambda x}{1 - \lambda^2} \, d\lambda = \frac{\pi}{2} \cos x, \text{ if } |x| < \frac{\pi}{2}$
8. $\int_{0}^{\pi} \frac{\lambda \sin \lambda x}{(\lambda^2 + \alpha^2)(\lambda^2 + \beta^2)} \, d\lambda = \frac{\pi}{2} \left( e^{-\alpha x} - e^{-\beta x} \right)$

9. Find Fourier sine integral representation of $f(x)$
   \[
f(x) = \begin{cases} 
0, & 0 < x < 1 \\
1, & 1 < x < 2 \\
0, & x > 2
\end{cases}
\]
   where $k$ is a constant.

10. Find the Fourier integral representation for the following functions:
   (i) $f(x) = \begin{cases} 
\frac{\pi}{2} \cos x, & |x| \leq \pi \\
0, & |x| > \pi
\end{cases}$
   (ii) $f(x) = \begin{cases} 
1, & |x| \leq \alpha \\
0, & |x| > \alpha
\end{cases}$
   (iii) $f(x) = e^{-|x|}, -\infty < x < \infty$. 

Answers

9. \( f(x) = \frac{2k}{\pi} \int_{0}^{\infty} \left( \frac{\cos \lambda x - \cos 2\lambda x}{\lambda} \right) \sin \lambda x \, d\lambda \)

10. (i) \( f(x) = \int_{0}^{\infty} \frac{\lambda \sin \lambda x}{1 - \lambda^2} \sin \lambda x \, d\lambda \)
   (ii) \( f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \lambda x \cos \lambda x}{\lambda} \, d\lambda \)
   (iii) \( f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{1 + \lambda^2} \cos \lambda x \, d\lambda \)

2.5. FOURIER TRANSFORMS AND INVERSION FORMULAE

(1) Fourier sine transform and its inversion formula

Fourier sine integral is

\[ f(x) = \frac{2}{\pi} \int_{0}^{\infty} f(t) \sin \lambda t \, dt \, d\lambda \]

Replacing \( \lambda \) by \( s \), we get

\[ f(x) = \frac{2}{\pi} \int_{0}^{\infty} f(t) \sin st \, dt \, ds \]

Denoting the value of the inner integral by \( F_s(s) \), we have

\[ f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_s(s) \sin sx \, ds \] (1)

where

\[ F_s(s) = \int_{0}^{\infty} f(x) \sin sx \, dx \] (2)

The function \( F_s(s) \) as defined by equation (2) is known as the Fourier sine transform of \( f(x) \) in \( 0 < x < \infty \).

The function \( f(x) \) as defined by equation (1) is called the inverse Fourier sine transform of \( F_s(s) \). Thus equation (1) gives the inversion formula for the Fourier sine transform.

Note. Some authors write the above formulae as:

\[ F_s(s) \text{ or } F_s[f(x)] = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} f(x) \sin sx \, dx \]

and

\[ f(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} F_s(s) \sin sx \, ds \] (3)

(2) Fourier cosine transform and its inversion formula

Fourier cosine integral is

\[ f(x) = \frac{2}{\pi} \int_{0}^{\infty} \cos \lambda x \, f(t) \cos \lambda t \, dt \, d\lambda \]

Replacing \( \lambda \) by \( s \), we get

\[ f(x) = \frac{2}{\pi} \int_{0}^{\infty} \cos sx \, f(t) \cos st \, dt \, ds \]

Denoting the value of the inner integral by \( F_c(s) \), we have

\[ f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_c(s) \cos sx \, ds \] (3)

where

\[ F_c(s) = \int_{0}^{\infty} f(x) \cos sx \, dx \] (4)

(3) Complex Fourier transform and its inversion formula (Anna 2007)

Complex form of Fourier integral is

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \int_{0}^{\infty} f(t) e^{i\lambda t} \, dt \, d\lambda \]

Replacing \( \lambda \) by \( s \), we get

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \int_{0}^{\infty} f(t) e^{ist} \, dt \, ds \]

Denoting the value of the inner integral by \( F(s) \), we have

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{i\alpha s} \, ds \] (5)

where

\[ F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} \, dx \] (6)

The function \( F(s) \) as defined by equation (6) is known as the Fourier transform of \( f(x) \).

The function \( f(x) \) as defined by equation (5) is called the inverse Fourier transform of \( F(s) \). Thus equation (5) gives the inversion formula for the Fourier transform.

Note. Some authors write the above formulae as:

\[ F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx \]

and

\[ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{isx} \, ds \] (7)

(4) Finite Fourier sine transform and its inversion formula

The finite Fourier sine transform of \( f(x) \) in \( 0 < x < c \) is defined as

\[ F_s(n) = \int_{0}^{c} f(x) \sin \frac{nx}{c} \, dx \]

where \( n \) is an integer.

The function \( f(x) \) is then called the inverse finite Fourier sine transform of \( F_s(n) \) and is given by

\[ f(x) = \frac{2}{c} \sum_{n=-\infty}^{\infty} F_s(n) \sin \frac{nx}{c} \]
(5) Finite Fourier cosine transform and its inversion formula

The finite Fourier cosine transform of \( f(x) \) in \( 0 < x < c \) is defined as

\[
F_c(n) = \int_0^c f(x) \cos \frac{nx}{c} \, dx
\]

where \( n \) is an integer.

The function \( f(x) \) is then called the inverse finite Fourier cosine transform of \( F_c(n) \) and is given by

\[
f(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^\infty F_c(n) \cos \frac{nx}{c}.
\]

ILLUSTRATIVE EXAMPLES

Example 1. Find the Fourier sine transform of \( e^{-|x|} \). Hence evaluate \( \int_0^\infty \frac{x \sin mx}{1+x^2} \, dx \).

(V.T.U., 2007)

Sol. In the interval \( (0, \infty) \), \( x \) is positive so that \( e^{-|x|} = e^{-x} \). Fourier sine transform of \( f(x) = e^{-x} \) is given by

\[
F_s(f(x)) = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty e^{-x} \sin sx \, dx
\]

\[
= \left[ \frac{e^{-x} (- \sin sx - s \cos sx)}{1+s^2} \right]_0^\infty = \frac{s}{1+s^2}
\]

Using inversion formula for Fourier sine transform, we get

\[
f(x) = \frac{2}{\pi} \int_0^\infty F_s(f(x)) \sin sx \, ds = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds
\]

Replacing \( x \) by \( m \), we have \( e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin ms \, ds = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{1+x^2} \, dx
\]

Hence,

\[
\int_0^\infty \frac{x \sin mx}{1+x^2} \, dx = \frac{\pi}{2} e^{-m}.
\]

Example 2. Find the Fourier sine transform of

\[
f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2
\end{cases}
\]

Sol. Fourier transform of \( f(x) \) is

\[
F_s(f(x)) = \int_0^2 f(x) \sin sx \, dx
\]

Example 3. Find the Fourier sine transform of \( \frac{e^{-ax}}{x} \).

(P.T.U. 2006; M.D.U. 2005)

Sol. Fourier sine transform of \( f(x) = \frac{e^{-ax}}{x} \) is

\[
F_s(f(x)) = \int_0^2 \frac{e^{-ax}}{x} \sin sx \, dx
\]

\[
= \int_0^2 \frac{e^{-ax} \sin sx}{x} \, dx
\]

Differentiating w.r.t. \( s \), we have

\[
\frac{dl}{ds} = \int_0^2 \frac{d}{ds} \left( \frac{e^{-ax}}{x} \sin sx \right) \, dx
\]

\[
= \int_0^2 \frac{e^{-ax}}{x} \cdot x \cos sx \, dx = \int_0^2 e^{-ax} \cos sx \, dx
\]

\[
= \left[ \frac{e^{-ax}}{a^2+s^2} \cdot (a \cos sx + s \sin sx) \right]_0^2
\]

\[
= 0 - \frac{1}{a^2+s^2} (-a) = \frac{a}{s^2+a^2}
\]

Integrating w.r.t. \( s \), we get

\[
I = \tan^{-1} \frac{s}{a} + c
\]

Now, when \( s = 0 \), from (1), \( I = 0 \)

\[
\therefore \quad \text{From (2), } 0 = 0 + c \quad \Rightarrow \quad c = 0
\]

\[
\therefore \quad I = \tan^{-1} \frac{s}{a} \quad \text{or} \quad F_s \left( \frac{e^{-ax}}{x} \right) = \tan^{-1} \frac{s}{a}.
\]
Example 4. Find Fourier sine transform of \( \frac{1}{x(x^2 + a^2)} \).

Sol. Fourier sine transform of \( f(x) = \frac{1}{x(x^2 + a^2)} \) is

\[
\mathcal{F}_s(f(x)) = \int_0^\infty f(x) \sin sx \, dx
\]

\[
= \int_0^\infty \frac{\sin sx}{x^2 + a^2} \, dx = \frac{\pi}{a} \tan^{-1} \frac{x}{a}
\]  (say) \( \ldots (1) \)

Differentiating w.r.t. \( s \), we have

\[
\frac{dI}{ds} = \int_0^\infty \frac{x \cos sx}{x^2 + a^2} \, dx = \int_0^\infty \frac{\cos sx}{x^2 + a^2} \, dx
\]

\( \ldots (2) \)

Differentiating again w.r.t. \( s \), we have

\[
\frac{d^2 I}{ds^2} = \int_0^\infty \frac{-x \sin sx}{x^2 + a^2} \, dx = \int_0^\infty \frac{-x^2 \sin sx}{x^2 + a^2} \, dx
\]

\[
= 2a^2 \int_0^\infty \frac{\sin sx}{x(x^2 + a^2)} \, dx - \int_0^\infty \frac{\sin sx}{x} \, dx = a^2 \pi - \frac{\pi}{2}
\]

\( \Rightarrow \)

\[
(D^2 - a^2) I = -\frac{\pi}{2}, \quad \text{where} \quad D = \frac{d}{ds}
\]

Its A.E. is

\[
D^2 - a^2 = 0 \quad \Rightarrow D = \pm a
\]

C.F. = \( c_1 e^{as} + c_2 e^{-as} \)

P.I. = \( \frac{1}{D^2 - a^2} \int \frac{\pi}{2} e^{as} \, ds = -\frac{\pi}{2} \frac{1}{D^2 - a^2} e^{as} = -\frac{\pi}{2} \frac{1}{2a} = \frac{\pi}{2a^2} \)

\( \Rightarrow \)

\[
I = \text{C.F.} + \text{P.I.}
\]

\[
I = c_1 e^{as} + c_2 e^{-as} + \frac{\pi}{2a^2}
\]

\( \ldots (3) \)

\[
\frac{dI}{ds} = ac_1 e^{as} - ac_2 e^{-as}
\]

\( \ldots (4) \)

When \( s = 0 \), from (1), \( I = 0 \)

From (3), \( I = c_1 + c_2 + \frac{\pi}{2a^2} \)

\[
\Rightarrow c_1 + c_2 + \frac{\pi}{2a^2} = 0
\]

\( \ldots (5) \)

When \( s = 0 \), from (2), \( \frac{dI}{ds} = 1 \frac{1}{a} \tan^{-1} \frac{x}{a} = \frac{\pi}{2a} \)

From (4),

\[
\frac{dI}{ds} = ac_1 - ac_2
\]

\( \Rightarrow \)

\[
ac_1 - ac_2 = \frac{\pi}{2a}
\]

or

\[
c_1 - c_2 = \frac{\pi}{2a^2} = 0
\]

\( \ldots (6) \)

Solving (5) and (6), \( c_1 = 0, \quad c_2 = -\frac{\pi}{2a^2} \)

\( \Rightarrow \)

\[
I = -\frac{\pi}{2a^2} e^{as} + \frac{\pi}{2a^2} (1 - e^{-as})
\]

Example 5. Find the Fourier sine transform of \( \frac{1}{x} \).

Sol. Fourier sine transform of \( \frac{1}{x} \) is given by

\[
\mathcal{F}_s \left( \frac{1}{x} \right) = \int_0^\infty \frac{1}{x} \sin sx \, dx
\]

Putting \( sx = \theta \) \( \Rightarrow \) \( x = \frac{\theta}{s} \), we have

\[
\mathcal{F}_s \left( \frac{1}{x} \right) = \int_0^\infty \frac{1}{s} \sin \theta \frac{d\theta}{s} = \frac{\pi}{2}
\]

Example 6. Find the Fourier sine and cosine transforms of \( x^{n-1}, \ n > 0 \).

Sol. We know that

\[
\mathcal{F}_s (x^{n-1}) = \int_0^\infty x^{n-1} \sin sx \, dx \quad \ldots (1)
\]

and

\[
\mathcal{F}_c (x^{n-1}) = \int_0^\infty x^{n-1} \cos sx \, dx \quad \ldots (2)
\]

Now,

\[
\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx, \ n > 0
\]

Putting \( t = ax, \ a > 0 \), we have

\[
\Gamma(n) = \int_0^\infty e^{-ax} (ax)^{n-1} \, dx = a^n \int_0^\infty e^{-ax} x^{n-1} \, dx
\]
\[ \int_0^{e^{-ax}} x^{n-1} \, dx = \frac{\Gamma(n)}{a^n} \]

Putting \( a = i \), we have

\[ \int_0^{e^{-ix}} x^{n-1} \, dx = \frac{\Gamma(n)}{(is)^n} \]

\[ \int_0^{\cos sx - i \sin sx} x^{n-1} \, dx = \left( \frac{i}{s} \right)^n \frac{\Gamma(n)}{s^n} = (-i)^n \frac{\Gamma(n)}{s^n} \]

\[ = \left( \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right)^n \frac{\Gamma(n)}{s^n} \]

\[ = \left( \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right)^n \frac{\Gamma(n)}{s^n} \]

Equating real and imaginary parts on both sides, we get

\[ \int_0^{\cos sx} \cos dx = \frac{\Gamma(n)}{s^n} \cos \frac{n\pi}{2} \]

and

\[ \int_0^{\sin sx} \sin dx = \frac{\Gamma(n)}{s^n} \sin \frac{n\pi}{2} \]

\[ \therefore \text{ From (1), } F_s(x^{n-1}) = \frac{\Gamma(n)}{s^n} \sin \frac{n\pi}{2} \]

and from (2),

\[ F_s(x^{n-1}) = \frac{\Gamma(n)}{s^n} \cos \frac{n\pi}{2} \]

Note. Taking \( n = \frac{1}{2} \), we get

\[ F_s \left( \frac{1}{\sqrt{s}} \right) = \frac{\Gamma \left( \frac{1}{2} \right)}{\sqrt{2\pi}} = \frac{\sqrt{\pi}}{\sqrt{2\pi}} = \frac{\sqrt{\pi}}{2\sqrt{s}} \]

\[ F_s \left( \frac{1}{\sqrt{s}} \right) = \frac{\Gamma \left( \frac{1}{2} \right)}{\sqrt{2\pi}} = \frac{\sqrt{\pi}}{2\sqrt{s}} \]

\[ \textbf{Example 7. Find the Fourier cosine transform of } e^{-x^2}. \]

\[ \textbf{Sol. Fourier cosine transform of } e^{-x^2} \text{ is given by} \]

\[ F_c(e^{-x^2}) = \int_0^\infty e^{-x^2} \cos sx \, dx = I \text{ (say)} \]

\[ \text{... (1)} \]

\[ \text{Differentiating w.r.t. } s, \text{ we have} \]

\[ \frac{di}{ds} = -\int_0^\infty e^{-x^2} \sin sx \, dx = \frac{1}{2} \int_0^\infty \sin ax (-2x e^{-x^2}) \, dx \]

\[ = \frac{1}{2} \left[ \sin ax e^{-x^2} \right]_0^\infty - \frac{x}{2} \int_0^\infty \cos ax e^{-x^2} \, dx \]

\[ = -\frac{s}{2} \int_0^\infty e^{-x^2} \cos ax \, dx = -\frac{s}{2} \]

\[ \text{or} \]

\[ \frac{di}{ds} = -\frac{s}{2} \]

Integrating, we have

\[ \log I = -\frac{s^2}{4} + \log A \text{ or } I = A e^{-\frac{s^2}{4}} \]

Now when \( s = 0 \), from (1),

\[ I = \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \]

\[ \therefore \text{ From (2), } \frac{\sqrt{\pi}}{2} = A \]

\[ \text{Hence } 1 = F_c(e^{-x^2}) = \frac{\sqrt{\pi}}{2} e^{-\frac{x^2}{4}}. \]

\[ \text{Note. } I = \int_0^\infty e^{-x^2} \, dx \text{ Put } x^2 = t \text{ i.e., } x = \sqrt{t} \]

\[ = \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2} \int_0^\infty t^{-1/2} e^{-t} \, dt = \frac{1}{2} \Gamma \left( \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2}. \]

\[ \textbf{Example 8. Find the Fourier cosine transform of} \]

\[ f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases} \]

\[ \textbf{Sol. Fourier cosine transform of } f(x) \text{ is} \]

\[ F_c(f(x)) = \int_0^\infty f(x) \cos sx \, dx = \int_0^a f(x) \cos sx \, dx + \int_a^\infty f(x) \cos sx \, dx \]

\[ = \int_0^a \cos x \cos sx \, dx + \int_0^\infty 0 \, dx = \frac{1}{2} \int_0^a 2 \cos x \cos sx \, dx + 0 \]

\[ = \frac{1}{2} \int_0^a [\cos(1+s)x + \cos(1-s)x] \, dx \]

\[ = \frac{1}{2} \left[ \frac{\sin(1+s)x}{1+s} + \frac{\sin(1-s)x}{1-s} \right]_0^a = \frac{1}{2} \left[ \frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s} \right] \]
Example 9. Find the Fourier cosine transform of \( f(x) = \frac{1}{1 + x^2} \). (M.D.U. May 2011)

Hence derive Fourier sine transform of \( g(x) = \frac{1}{x(1 + x^2)} \).

Sol. Fourier cosine transform of \( f(x) = \frac{1}{1 + x^2} \) is

\[ F_c(f(x)) = \int_0^\infty \frac{\cos sx}{1 + x^2} \, dx = 1 \quad \text{(say)} \quad \text{... (1)} \]

Differentiating w.r.t. \( s \), we have

\[
\frac{dF}{ds} = \int_0^\infty \frac{-x \sin sx}{1 + x^2} \, dx = \int_0^\infty \frac{-x^2 \sin sx}{x(1 + x^2)} \, dx
\]

\[ = \int_0^\infty \left[ \frac{1}{1 + x^2} \right] \sin sx \, dx = \int_0^\infty \frac{\sin sx}{x(1 + x^2)} \, dx - \int_0^\infty \frac{\sin sx}{x} \, dx \]

\[ = \int_0^\infty \frac{-\sin sx}{x(1 + x^2)} \, dx - \frac{\pi}{2} \]

... (2)

Differentiating again w.r.t. \( s \), we have

\[
\frac{d^2F}{ds^2} = \int_0^\infty \frac{x \cos sx}{(1 + x^2)^2} \, dx = \int_0^\infty \frac{\cos sx}{1 + x^2} \, dx = 1
\]

\[ \Rightarrow \quad (D^2 - 1) I = 0, \quad \text{where} \quad D = \frac{d}{ds} \]

Its A.E. is \( D^2 - 1 = 0 \) whence \( D = 1 \)

\[ \Rightarrow \quad I = c_1 e^s + c_2 e^{-s} \]

\[ \frac{dI}{ds} = c_1 e^s - c_2 e^{-s} \quad \text{... (4)} \]

When \( s = 0 \), from (1), \( I = \int_0^\infty \frac{dx}{1 + x^2} = \left[ \tan^{-1} x \right]_0^\infty = \frac{\pi}{2} \)

From (3), \( I = c_1 + c_2 \)

\[ \Rightarrow \quad c_1 + c_2 = \frac{\pi}{2} \quad \text{... (5)} \]

When \( s = 0 \), from (2), \( \frac{dI}{ds} = \frac{\pi}{2} \)

From (4), \( \frac{dI}{ds} = c_1 - c_2 \)

\[ \Rightarrow \quad c_1 - c_2 = \frac{\pi}{2} \quad \text{... (6)} \]

Solving (5) and (6), \( c_1 = 0, \quad c_2 = \frac{\pi}{2} \)

\[ \therefore \quad I = \frac{\pi}{2} e^{-s} \Rightarrow F_s(f(x)) = \frac{\pi}{2} e^{-s} \]

Putting the values of \( c_1 \) and \( c_2 \) in (4)

\[ \frac{dI}{ds} = \frac{\pi}{2} e^{-s} \]

\[ \therefore \quad \text{From (2),} \quad \frac{-\pi}{2} e^{-s} = \int_0^\infty \frac{\sin sx}{x(1 + x^2)} \, dx - \frac{\pi}{2} \]

\[ \Rightarrow \quad \int_0^\infty \frac{\sin sx}{x(1 + x^2)} \, dx = \frac{\pi}{2} \left( 1 - e^{-s} \right) \]

\[ \Rightarrow \quad F_s(g(x)) = \frac{\pi}{2} (1 - e^{-s}) \]

Note. In example 9 above, we have proved that

\[ \int_0^\infty \frac{\sin sx}{x(1 + x^2)} \, dx = \frac{\pi}{2} e^{-s} \]

Differentiating w.r.t. \( s \), we get

\[ \int_0^\infty \frac{\frac{d}{ds} \left( \frac{\cos sx}{1 + x^2} \right)}{x} \, dx = \frac{\pi}{2} e^{-s} \]

\[ \Rightarrow \quad \int_0^\infty \frac{-x \sin sx}{1 + x^2} \, dx = \frac{\pi}{2} e^{-s} \]

\[ \Rightarrow \quad \int_0^\infty \frac{x}{1 + x^2} \sin sx \, dx = \frac{\pi}{2} e^{-s} \]

\[ \Rightarrow \quad F_s(g(x)) = \frac{\pi}{2} e^{-s} \]

\[ \Rightarrow \quad \text{Fourier sine transform of} \quad \frac{x}{1 + x^2} \text{is} \quad \frac{\pi}{2} e^{-s}. \]

Example 10. Find the Fourier transform of

\[ f(x) = \begin{cases} \frac{1}{1 + x^2} & |x| < a \\ 0 & |x| > a \end{cases} \quad \text{(M.D.U. May 2011; Anna 2007)} \]

Hence evaluate

\[ (i) \quad \int_0^a \frac{\sin sx}{s} \, dx \quad \text{and} \quad \int_0^a \frac{\sin x}{x} \, dx = \frac{\pi}{2} \]
Sol. Fourier transform of \( f(x) \) is given by

\[
F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, dx
\]

\[
= \int_{-\infty}^{0} f(x) e^{i\alpha x} \, dx + \int_{0}^{\infty} f(x) e^{i\alpha x} \, dx
\]

\[
= \int_{-\infty}^{0} 0 \cdot e^{i\alpha x} \, dx + \int_{0}^{\infty} 1 \cdot e^{i\alpha x} \, dx
\]

\[
= \left[ \frac{e^{i\alpha x}}{i\alpha} \right]_{-\infty}^{0} = \frac{e^{i\alpha 0} - e^{-i\alpha \infty}}{i\alpha} = 2 \sin \frac{\alpha s}{s}
\]

For \( s = 0 \), we find \( F(f(x)) = 2\pi \alpha \)

By inversion formula for Fourier transform, we have

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(f(x)) \cdot e^{-i\alpha x} \, dx
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin \frac{\alpha s}{s} \cdot e^{-i\alpha x}}{s} \, ds
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{\alpha s}{s} \cos \alpha x - i \sin \alpha x}{s} \, ds
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{\alpha s}{s} \cos \alpha x}{s} \, ds - i \frac{\sin \alpha x}{s} \int_{-\infty}^{\infty} \frac{\alpha s}{s} \, ds
\]

\[= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{\alpha s}{s} \cos \alpha x}{s} \, ds \]

[Second integral vanishes since the integrand is an odd function of \( s \)]

\[\int_{-\infty}^{\infty} \frac{\sin \frac{\alpha s}{s} \cos \alpha x}{s} \, ds = \pi f(x) = \begin{cases} \pi, & |x| < \alpha \\ 0, & |x| > \alpha \end{cases} \] \quad \ldots (1)

Since the integrand in (1) is an even function of \( s \), we have

\[2 \int_{0}^{\infty} \frac{\sin \frac{\alpha s}{s} \cos \alpha x}{s} \, ds = \pi f(x) = \begin{cases} \pi, & |x| < \alpha \\ 0, & |x| > \alpha \end{cases} \]

\[\int_{0}^{\infty} \frac{\sin \frac{\alpha s}{s} \cos \alpha x}{s} \, ds = \frac{\pi}{2}, \quad |x| < \alpha \]

\[\int_{0}^{\infty} \frac{\sin \frac{\alpha s}{s} \cos \alpha x}{s} \, ds = 0, \quad |x| > \alpha \]

Example 11. Find the Fourier transform of \( f(x) = \begin{cases} 1 - x^2, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases} \)

and use it to evaluate \( \int_{0}^{\infty} \left( \frac{x \cos x - \sin x}{x^3} \right) \cos \frac{x}{2} \, dx \).


Sol. Fourier transform of \( f(x) \) is given by

\[
F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, dx
\]

\[
= \int_{-\infty}^{1} f(x) e^{i\alpha x} \, dx + \int_{1}^{\infty} f(x) e^{i\alpha x} \, dx + \int_{-\infty}^{0} f(x) e^{i\alpha x} \, dx + \int_{0}^{1} f(x) e^{i\alpha x} \, dx
\]

\[
= 0 \cdot dx + \int_{1}^{\infty} (1-x^2) e^{i\alpha x} \, dx + \int_{0}^{1} 0 \cdot dx + \int_{-\infty}^{1} 0 \cdot dx
\]

\[
= \left[ \frac{e^{i\alpha x}}{i\alpha} \right]_{-\infty}^{1} = \frac{e^{i\alpha 1} - e^{-i\alpha \infty}}{i\alpha} = \frac{4}{i\alpha} \left( e^{i\alpha 1} - e^{-i\alpha 1} \right)
\]

\[
= \left[ \frac{4}{i\alpha} \left( e^{i\alpha 1} - e^{-i\alpha 1} \right) \right]_{-\infty}^{\infty} = 4 \left( \frac{\cos \frac{\alpha s}{s} - \sin \frac{\alpha s}{s}}{s^3} \right)
\]

By inversion formula for Fourier transform, we have

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(f(x)) \cdot e^{-i\alpha x} \, dx
\]

\[
= -\frac{2}{\pi} \int_{0}^{\infty} \left( \frac{\cos \frac{\alpha s}{s} - \sin \frac{\alpha s}{s}}{s^3} \right) \cos \frac{x}{2} \, ds
\]
\[ -\frac{2}{\pi} \int_0^\infty \left( \frac{s \cos s \sin s}{s^3} \right) \cos sx \, ds + \frac{2i}{\pi} \int_{-\infty}^0 \left( \frac{s \cos s \sin s}{s^3} \right) \sin sx \, ds \]
\[ = -\frac{4}{\pi} \int_0^\infty \left( \frac{s \cos s \sin s}{s^3} \right) \cos sx \, ds \]

(Since the integrand in the first integral is even and that in the second integral is odd)

\[ \int_0^\infty \left( \frac{s \cos s \sin s}{s^3} \right) \cos sx \, ds = -\frac{\pi}{4} f(x) = \begin{cases} \frac{\pi}{4} (1-x^2), & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases} \]

Putting \( x = \frac{1}{2} \), we have \( \int_0^\infty \left( \frac{s \cos s \sin s}{s^3} \right) \cos \frac{s}{2} \, ds = -\frac{\pi}{4} \left( 1 - \frac{1}{4} \right) = -\frac{3\pi}{16} \)

Hence \( \int_0^\infty \left( \frac{x \cos x - \sin x}{x^2} \right) \cos \frac{x}{2} \, dx = -\frac{3\pi}{16} \).

Example 12. Find the inverse Fourier transform of \( F(a) = e^{-|x|} \).

Sol. The inverse Fourier transform of \( F(a) = e^{-|b|} \) is given by

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty F(a) e^{ixa} \, da = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-|b|} e^{ixa} \, da \]

\[ = \frac{1}{2\pi} \left[ \int_{-\infty}^{0} e^{iax} e^{-ix} \, dx + \int_{0}^{\infty} e^{iax} e^{ix} \, dx \right] \]

\[ = \frac{1}{2\pi} \left[ \int_{-\infty}^{0} e^{i(ax-x)} \, dx + \int_{0}^{\infty} e^{i(ax+x)} \, dx \right] \]

\[ = \frac{1}{2\pi} \left[ \frac{e^{i(y-x)}}{y-x} \right]_{-\infty}^{0} + \frac{e^{i(y+x)}}{y+x} \right]_{0}^{\infty} \]

\[ = \frac{1}{2\pi} \left[ \frac{1}{y-ix} + \frac{1}{y+ix} \right] = \frac{1}{2\pi} \left( \frac{2y}{y^2 + x^2} \right) \]

\[ = \frac{y}{\pi (y^2 + x^2)}. \]

Example 13. Solve the integral equation \( \int_0^l f(x) \cos px \, dx = \begin{cases} 1-p, & 0 \leq p \leq 1 \\ 0, & p > 1 \end{cases} \)

Hence deduce that \( \int_0^\infty \frac{\sin^2 x}{x^2} \, dx = \frac{\pi}{2}. \)

Sol. Here \( \int_0^l f(x) \cos px \, dx = F_p(p) \)

\[ F_p(p) = \begin{cases} 1-p, & 0 \leq p \leq 1 \\ 0, & p > 1 \end{cases} \]

Example 14. Find the finite Fourier sine transform of \( f(x) = \begin{cases} \frac{2k}{l}, & 0 \leq x \leq \frac{l}{2} \\ \frac{2k}{l} (l-x), & \frac{l}{2} \leq x \leq l \end{cases} \)

Sol. Finite Fourier sine transform of \( f(x) \) in \( 0 \leq x \leq l \) is

\[ F_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \]

\[ = \int_0^{l/2} f(x) \sin \frac{n\pi x}{l} \, dx + \int_{l/2}^l f(x) \sin \frac{n\pi x}{l} \, dx \]

\[ = \int_0^{l/2} \frac{2k}{l} x \sin \frac{n\pi x}{l} \, dx + \int_{l/2}^l \frac{2k}{l} (l-x) \sin \frac{n\pi x}{l} \, dx \]

\[ = \frac{2k}{l} \int_0^{l/2} x \sin \frac{n\pi x}{l} \, dx + \frac{2k}{l} \int_{l/2}^l (l-x) \sin \frac{n\pi x}{l} \, dx \]

(Integrating by parts)

\[ = \frac{2k}{l} \left[ -\cos \frac{n\pi x}{l} + \frac{n\pi x}{l} \sin \frac{n\pi x}{l} \right]_0^{l/2} + \frac{2k}{l} \left[ (l-x) \cos \frac{n\pi x}{l} - (-1) \sin \frac{n\pi x}{l} \right]_{l/2}^l \]
Example 15. Find the finite Fourier cosine transform of
\[ f(x) = \left(1 - \frac{x^2}{\pi^2}\right), \quad 0 \leq x < \pi. \]

Sol. Finite Fourier cosine transform of \( f(x) \) in \( 0 \leq x \leq \pi \) is
\[ F_c(n) = \int_0^\pi f(x) \cos nx \, dx = \int_0^\pi \left(1 - \frac{x^2}{\pi^2}\right) \cos nx \, dx \]
(Integrating by parts)
\[ \quad \quad = \left[ \frac{(1 - x^2)}{n^2} \sin nx \right]_0^\pi - 2 \int_0^\pi \left(1 - \frac{x^2}{\pi^2}\right) \frac{\cos nx}{n} \, dx. \]
\[ \quad \quad = \frac{2}{n^2} \sin \pi n \quad \text{when } n \neq 0. \]

If \( n = 0 \), then \( F_c(0) = \int_0^\pi \left(1 - \frac{x^2}{\pi^2}\right) \, dx \]
\[ = \left[ \frac{x \left(1 - \frac{x^2}{\pi^2}\right)}{\pi^2} \right]_0^\pi = -\frac{\pi}{2} (1 - 0) = \frac{\pi}{2} \]
\[ \therefore \quad F_c(n) = \begin{cases} \frac{2}{n^2}, & \text{if } n = 1, 2, 3, \ldots \\ \frac{\pi}{3}, & \text{if } n = 0. \end{cases} \]

Example 16. Find \( f(x) \), if \( F_c(n) = \frac{1 - \cos \frac{n\pi}{2}}{n^2 \pi^2} \), where \( 0 \leq x \leq \pi \).

Sol. Here \( F_c(n) = \frac{1 - \cos \frac{n\pi}{2}}{n^2 \pi^2} \) is the finite Fourier sine transform of \( f(x) \) in \( 0 \leq x \leq \pi \).
\[ f(x) = \text{inverse finite Fourier sine transform of } F_c(n) \]
\[ \quad = \sum_{n=1}^\infty \frac{F_c(n)}{c} \sin \frac{n\pi x}{c} = \sum_{n=1}^\infty \left(\frac{1 - \cos \frac{n\pi}{2}}{n^2 \pi^2}\right) \sin \frac{n\pi x}{c} \quad (\because \text{here } c = \pi) \]
\[ \quad = \sum_{n=1}^\infty \left(\frac{1 - \cos \frac{n\pi}{2}}{n^2 \pi^2}\right) \sin nx. \]
(iii) Find the Fourier transform of the function

\[ f(t) = \begin{cases} 
1, & \text{for } -2 < t < 1 \\
2, & \text{for } -1 < t < 1 \\
1, & \text{for } 1 < t < 2 \\
0, & \text{otherwise}
\end{cases} \]

7. Using inverse Fourier sine transform, find \( f(x) \) if

(i) \( F_{y}(y) = \frac{1}{y} e^{-a} \) \( (M.D.U. 2009) \)

(ii) \( F_{y}(y) = \frac{y}{1 + y^2} \)

8. Find the finite Fourier sine and cosine transforms of the following functions:

(i) \( f(x) = 2x, \ 0 \leq x \leq 4 \)

(ii) \( f(x) = x(l-x), \ 0 \leq x \leq l \)

(iii) \( f(x) = x^2, \ 0 \leq x \leq 2 \)

(iii) \( f(x) = a \left( 1 - \frac{x}{l} \right), \ 0 \leq x \leq l \)

9. Find the finite Fourier sine transform of the following functions:

(i) \( f(x) = \cos x, \ 0 \leq x \leq \pi \)

(ii) \( f(x) = \frac{2x}{3}, \ 0 \leq x \leq \frac{\pi}{3} \)

(iii) \( f(x) = \frac{\pi - x}{3}, \ \frac{\pi}{3} \leq x \leq \pi \)

10. If \( f(x) = \sin ax \), where \( 0 \leq x \leq \pi \) and \( a > 0 \) is a positive integer, show that

\[ F_{a}(n) = \begin{cases} 
0, & \text{if } n = k \\
\frac{\pi}{2}, & \text{if } n = k
\end{cases} \]

11. Find the finite Fourier cosine transform of the following functions:

(i) \( f(x) = \sin x, \ 0 \leq x \leq \pi \)

(ii) \( f(x) = \frac{2x}{3}, \ 0 \leq x \leq \frac{\pi}{3} \)

(iii) \( f(x) = \frac{\pi - x}{3}, \ \frac{\pi}{3} \leq x \leq \pi \)

12. Find \( f(x) \), if

(i) \( F_{a}(n) = \frac{2}{n} \sin \frac{\pi n}{4}, \ 0 \leq n \leq 1 \)

(ii) \( F_{a}(n) = \frac{1}{n^2} \sin \frac{\pi n}{3}, \ 0 \leq n \leq 1 \)

(iii) \( F_{a}(n) = \frac{2a^3}{n^3} (1 - \cos n\pi) \), \( 0 \leq n \leq 1 \)

(iv) \( F_{a}(n) = \frac{\pi n a}{l} \) and \( F_{(l-x)}(0) = ac \), where \( 0 \leq x \leq l \)

(v) \( F_{a}(n) = -\frac{b}{n^2} (1 + \cos n\pi) \) and \( F_{(l-x)}(0) = \frac{b^3}{6} \), where \( 0 \leq x \leq l \)

(vi) \( F_{a}(n) = \frac{\cos \left( \frac{2\pi n}{3} \right)}{(2n+1)^3} \), where \( 0 \leq n \leq 1 \).

13. Solve the following integral equations:

(i) \( \int_{0}^{l} f(x) \sin\lambda x \ dx = \frac{1}{2}, \ \lambda < 1 \)

(ii) \( \int_{0}^{l} f(x) \cos\lambda x \ dx = e^\lambda, \ \lambda > 0 \)

(iii) \( \int_{0}^{l} f(x) \sin\lambda x \ dx \)

Answers

(i) \( a \cos a - b \cos ab + \sin ab - \sin a \)

(ii) \( \frac{n}{\sqrt{2}} \)

(iii) \( 10 \left( \frac{1}{s^2 + 25} + \frac{1}{s^2 + 4} \right) \)

(iv) \( \frac{3}{ns^3} \)

(v) \( \frac{1}{s^2 + a^2 + s^2 + a^2} \)

(vi) \( \frac{2}{s^3} \)

(vii) \( \frac{1}{s^2 + 2} \sin a + 2a \cos a \sin a \)

(viii) \( \frac{2}{s^2} \)

(ix) \( \frac{2}{1 + s^2} \)

(x) \( \frac{1}{s} \) \( e^{\alpha s - e^{\beta s}} \)

(xi) \( \frac{2}{s} \)

(xii) \( \frac{1}{s} \) \( 1 - \cos a \)

(xiii) \( \frac{1 + s^2}{s} \)

(xiv) \( \frac{1}{s^2 + 1} \) \( e^{\alpha s - e^{\beta s}} \)

(xv) \( \frac{2}{ns^3} \)

(xvi) \( \frac{1}{s^2 + 2} \sin a + 2a \cos a \sin a \)

(xvii) \( \frac{2}{s^2} \)

(xviii) \( \frac{1}{s} \) \( 1 - \cos a \)

(xix) \( \frac{2}{s^2} \)

(xx) \( \frac{1}{s^2 + 1} \) \( e^{\alpha s - e^{\beta s}} \)

(xxi) \( \frac{2}{ns^3} \)

(xxii) \( \frac{1}{s^2 + 2} \sin a + 2a \cos a \sin a \)
9. \( \frac{n}{n^2-1} \left[ 1 + (-1)^n \right] \)

11. \( \frac{1}{n^2} \sin \frac{n\pi}{3} \)

12. \( \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{l} \)

2.6. PROPERTIES OF FOURIER TRANSFORMS

1. Linearity Property. If \( F(s) \) and \( G(s) \) are Fourier transforms of \( f(x) \) and \( g(x) \) respectively, then

\[ F(a f(x) + b g(x)) = a F(s) + b G(s) \]

where \( a \) and \( b \) are constants.

Proof. By definition of Fourier transform, we have

\[ F(s) = F(f(x)) = \int_{-\infty}^{\infty} f(x) \cdot e^{i\omega x} \, dx \]

and

\[ G(s) = F(g(x)) = \int_{-\infty}^{\infty} g(x) \cdot e^{i\omega x} \, dx \]

\[ F[a f(x) + b g(x)] = \int_{-\infty}^{\infty} [a f(x) + b g(x)] \cdot e^{i\omega x} \, dx \]

\[ = a \int_{-\infty}^{\infty} f(x) \cdot e^{i\omega x} \, dx + b \int_{-\infty}^{\infty} g(x) \cdot e^{i\omega x} \, dx \]

\[ = a F(s) + b G(s). \]

Cor. (i) If \( F_s(s) \) and \( G_s(s) \) are the Fourier sine transforms of \( f(x) \) and \( g(x) \) respectively, then

\[ F_s[a f(x) + b g(x)] = a F_s(s) + b G_s(s) \]

where \( a \) and \( b \) are constants.

(ii) If \( F_c(s) \) and \( G_c(s) \) are the Fourier cosine transforms of \( f(x) \) and \( g(x) \) respectively, then

\[ F_c[a f(x) + b g(x)] = a F_c(s) + b G_c(s) \]

where \( a \) and \( b \) are constants.

2. Change of scale property (Similarity Theorem). If \( F(s) \) is the complex Fourier transform of \( f(x) \), then

\[ F(f(ax)) = \frac{1}{a} F\left( \frac{s}{a} \right), \quad a \neq 0. \]

Proof. By definition of complex Fourier transform, we have

\[ F(s) = F(f(x)) = \int_{-\infty}^{\infty} f(x) \cdot e^{i\omega x} \, dx \]

Putting \( x = t \) \( i.e., \), \( x = \frac{t}{a} \), we have \( dx = \frac{dt}{a} \).

When \( x \to -\infty, \, t \to -\infty \) and when \( x \to \infty, \, t \to \infty \).

\[ \therefore F(f(ax)) = \int_{-\infty}^{\infty} f(at) \cdot e^{i\omega t} \, dt = \frac{1}{a} \int_{-\infty}^{\infty} f(t) \cdot e^{i\omega \frac{t}{a}} \, dt = \frac{1}{a} F\left( \frac{\omega}{a} \right) \]

Cor. (i) If \( F_s(s) \) is the Fourier sine transform of \( f(x) \), then

\[ F_s(f(ax)) = \frac{1}{a} F_s\left( \frac{s}{a} \right) \]

(ii) If \( F_c(s) \) is the Fourier cosine transform of \( f(x) \), then

\[ F_c(f(ax)) = \frac{1}{a} F_c\left( \frac{s}{a} \right) \]

3. Shifting Property. If \( F(s) \) is the complex Fourier transform of \( f(x) \), then

\[ F(f(x-a)) = e^{i\omega a} F(s) \]

Proof. By definition of complex Fourier transform, we have

\[ F(s) = F(f(x)) = \int_{-\infty}^{\infty} f(x) \cdot e^{i\omega x} \, dx \]

Putting \( x-a = t \) \( i.e., \), \( x = a + t \), we have \( dx = dt \).

When \( x \to -\infty, \, t \to -\infty \) and when \( x \to \infty, \, t \to \infty \).

\[ \therefore F(f(x-a)) = \int_{-\infty}^{\infty} f(t) \cdot e^{i\omega (a+t)} \, dt = \int_{-\infty}^{\infty} f(t) \cdot e^{i\omega a} \cdot e^{i\omega t} \, dt \]

\[ = e^{i\omega a} \int_{-\infty}^{\infty} f(t) \cdot e^{i\omega t} \, dt = e^{i\omega a} F(s). \]
3. (a) Shifting on time axis. If $F(s)$ is the complex Fourier transform of $f(t)$ and $t_0$ is any real number, then

$$F(t-t_0) = e^{ist_0} F(s).$$

(M.D.U. May 2009)

Proof. By definition of complex Fourier transform, we have

$$F(s) = F(t) = \int f(t) e^{ist} dt$$

$$\therefore \quad F(t-t_0) = \int f(t-t_0) e^{ist} dt \tag{1}$$

Putting $t-t_0 = T$, i.e., $t = t_0 + T$, we have $dt = dT$

When $t \to -\infty$, $T \to -\infty$ and when $t \to \infty$, $T \to \infty$

$$\therefore \quad F(t-t_0) = \int f(T) e^{ist} dT = e^{ist_0} \int f(T) e^{ist} dT = e^{ist_0} F(s).$$

Remark. Inverse Fourier transform of $e^{ist_0}$ is $f(t)$.

3. (b) Shifting on frequency axis. If $F(s)$ is the complex Fourier transform of $f(t)$, and $s_0$ is any real number, then

$$F[e^{is_0} f(t)] = F(s + s_0).$$

(M.D.U. May 2009)

Proof. By definition of complex Fourier transform, we have

$$F(s) = F(t) = \int f(t) e^{ist} dt$$

$$\therefore \quad F[e^{is_0} f(t)] = \int e^{is_0} f(t) e^{ist} dt$$

$$= \int f(t) e^{is(t+s_0)} dt = F(s + s_0).$$

Remark. Inverse Fourier transform of $F(s + s_0)$ is $e^{is_0} f(t)$.

4. Modulation Theorem. If $F(s)$ is the complex Fourier transform of $f(t)$, then

$$F(f(t) \cos ax) = \frac{1}{2} [F(s + a) + F(s - a)].$$

Proof. By definition of complex Fourier transform, we have

$$F(s) = F(f(x)) = \int f(x) e^{ist} dx$$

$$\therefore \quad F[f(x) \cos ax] = \int f(x) \cos ax \cdot e^{ist} dx$$

$$= \int f(x) \left( e^{i(ax + \pi) / 2} \right) e^{ist} dx$$

$$= \frac{1}{2} \left( \int f(x) e^{i(ax + \pi) / 2} dx + \int f(x) e^{i(ax - \pi) / 2} dx \right)$$

$$= \frac{1}{2} \left( F(s + a) + F(s - a) \right).$$

Cor. If $F_c(s)$ and $F_s(s)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then

(i) $F_c[f(x) \cos ax] = \frac{1}{2} [F_c(s + a) + F_c(s - a)]$

(ii) $F_s[f(x) \sin ax] = \frac{1}{2} [F_s(s - a) - F_s(s + a)]$

(iii) $F_c[f(x) \cos ax] = \frac{1}{2} [F_c(s + a) + F_c(s - a)]$

(iv) $F_s[f(x) \sin ax] = \frac{1}{2} [F_s(s + a) - F_s(s - a)].$

5. If $F_c(s)$ and $F_s(s)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then

(i) $F_c[f(x) \cos ax] = -\frac{d}{ds} [F_s(s)]$

(ii) $F_s[f(x) \cos ax] = \frac{d}{ds} [F_c(s)]$

Proof. (i) $\frac{d}{ds} [F(s)] = \int f(s) \cos ax dx$

$$= \int f(x) (-x \sin ax) dx$$

$$= -\int f(x) \sin ax dx$$

$$= -F_s[f(x)]$$

$$\therefore \quad F_c[f(x) \cos ax] = -\frac{d}{ds} [F_s(s)]$

(ii) $\frac{d}{ds} [F_s(s)] = \int f(s) \sin ax dx$

$$= \int f(x) (x \cos ax) dx$$
\[ \int_0^1 (x f(x)) \cos sx \, dx \]
\[ = F_s \{ x f(x) \} \]
\[ = \frac{d}{ds} \{ F_s \{ s \} \} \]

Example 1. Find the Fourier transform of \( e^{-ax^2} \). Hence find the Fourier transform of

(i) \( e^{-ax^2} \) \( (a > 0) \) \, (M.D.U. May 2009)
(ii) \( e^{-\frac{x^2}{2}} \)
(iii) \( e^{-i4x-2x^2} \)
(iv) \( e^{-x^2} \cos 2x \)

Sol. Fourier transform of \( f(x) = e^{-x^2} \) is given by

\[ F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, dx = \int_{-\infty}^{\infty} e^{-x^2} e^{i\alpha x} \, dx \]
\[ = \int_{-\infty}^{\infty} e^{-(x^2 - i\alpha x)} \, dx = \int_{-\infty}^{\infty} e^{-\left( x - \frac{i\alpha}{2} \right)^2} dx \]
\[ = e^{-\frac{\alpha^2}{4}} \int_{-\infty}^{\infty} e^{-x^2} \, dx \]
\[ = e^{-\frac{\alpha^2}{4}} \sqrt{\pi} \]
\[ = \sqrt{\pi} e^{-\frac{\alpha^2}{4}} = F(e) \]

(i) \( e^{-ax^2} = e^{-\left(\sqrt{a}x\right)^2} = f(\sqrt{a}x) \)

By change of scale property, we have

\[ F(f(\sqrt{a}x)) = \frac{1}{\sqrt{a}} F \left( \frac{s}{\sqrt{a}} \right) \]
\[ \Rightarrow F(e^{-ax^2}) = \frac{1}{\sqrt{a}} \sqrt{\pi} e^{-\frac{1}{4a} \left( \frac{s}{\sqrt{a}} \right)^2} = \sqrt{\pi} e^{-\frac{s^2}{4a}} \]

(ii) Putting \( a = \frac{1}{2} \) in deduction (i), we have

\[ F \left( e^{-\frac{x^2}{2}} \right) = \sqrt{2\pi} e^{-\frac{s^2}{4}} \]

Example 2. Find the Fourier sine and cosine transform of \( x e^{-ax} \). (Madras 2006)

Sol. Let us first find the Fourier sine and cosine transforms of \( e^{-ax} \).

\[ F_s \{ e^{-ax} \} = \int_{-\infty}^{\infty} e^{-ax} \sin sx \, dx = \frac{e^{-ax}}{a^2 + s^2}(-a \sin sx + s \cos sx) \bigg|_{0}^{\infty} = \frac{s}{a^2 + s^2} \]
and

\[ F_c \{ e^{-ax} \} = \int_{-\infty}^{\infty} e^{-ax} \cos sx \, dx = \frac{e^{-ax}}{a^2 + s^2}(-a \cos sx + s \sin sx) \bigg|_{0}^{\infty} = \frac{a}{a^2 + s^2} \]

and

\[ F_s \{ xe^{-ax} \} = \frac{d}{ds} \left( F_c \{ e^{-ax} \} \right) = \frac{d}{ds} \left( \frac{a}{a^2 + s^2} \right) = \frac{2as}{(a^2 + s^2)^2} \]
and

\[ F_c \{ xe^{-ax} \} = \frac{d}{dx} \left( F_s \{ e^{-ax} \} \right) = \frac{d}{dx} \left( \frac{s}{a^2 + s^2} \right) = \frac{(a^2 + s^2)^2}{(a^2 + s^2)^2} = \frac{a^2 - s^2}{(a^2 + s^2)^2} \]

2.7 CONVOLUTION

The convolution of two functions \( f(x) \) and \( g(x) \) over the interval \((-\infty, \infty)\) is defined as

\[ f(x) * g(x) = \int_{-\infty}^{\infty} f(u) g(x - u) \, du \]
2.8. CONVOLUTION THEOREM FOR FOURIER TRANSFORMS
(or Faltung Theorem)

The Fourier transform of the convolution of \( f(x) \) and \( g(x) \) is equal to the product of the Fourier transforms of \( f(x) \) and \( g(x) \), i.e.,

\[
F[f(x) * g(x)] = F[f(x)] \cdot F[g(x)]
\]

Proof. By definition of Fourier transform,

\[
F[f(x)] = \int_{-\infty}^{\infty} f(x) \cdot e^{i\omega x} \, dx,
\]

\[
G(s) = F[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot e^{i\omega x} \, dx
\]

and by definition of convolution,

\[
f(x) * g(x) = \int_{-\infty}^{\infty} f(u) \cdot g(x-u) \, du
\]

\[
F[f(x) * g(x)] = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(u) \cdot g(x-u) \, du \right] e^{i\omega x} \, dx
\]

(By changing the order of integration)

Putting \( x-u = t \), i.e., \( x = u + t \), we have \( dx = dt \).

When \( x \to -\infty \), \( t \to -\infty \) and when \( x \to \infty \), \( t \to \infty \).

\[
F[f(x) * g(x)] = \int_{-\infty}^{\infty} f(u) \left[ \int_{-\infty}^{\infty} g(t) \cdot e^{i\omega(u+t)} \, dt \right] du
\]

= \[
\int_{-\infty}^{\infty} f(u) \int_{-\infty}^{\infty} g(t) \cdot e^{i\omega u} \cdot e^{i\omega t} \, dt \, du
\]

= \[
\int_{-\infty}^{\infty} f(u) \left[ \int_{-\infty}^{\infty} g(t) \cdot e^{i\omega t} \, dt \right] e^{i\omega u} \, du
\]

= \[
F[f(u)] \cdot F[g(x)] = F[f(x) \cdot g(x)]
\]

Remark. The following properties of convolution can be easily proved.

(i) \( f(x) * g(x) = g(x) \cdot f(x) \)

(ii) \( f(x) * [g(x) \cdot h(x)] = [f(x) * g(x)] \cdot h(x) \)

(iii) \( f(x) * [g(x) \cdot h(x)] = f(x) * g(x) + f(x) * h(x) \)

2.9. RELATION BETWEEN FOURIER AND LAPLACE TRANSFORMS

(\( P.T.U. \) 2005)

If

\[
f(t) = \begin{cases} 
    e^{-\alpha t} g(t), & t > 0 \\
    0, & t < 0
\end{cases}
\]

then \( F[f(t)] = L[g(t)] \).

Proof. By definition of Fourier transform, we have

\[
F[f(t)] = \int_{-\infty}^{\infty} f(t) \cdot e^{i\omega t} \, dt
\]

= \[
\int_{0}^{\infty} e^{-\alpha t} g(t) \cdot e^{i\omega t} \, dt + \int_{-\infty}^{0} e^{-\alpha t} g(t) \cdot e^{i\omega t} \, dt
\]

= \[
\int_{0}^{\infty} e^{-(\alpha - i\omega) t} g(t) \, dt
\]

= \[
\int_{0}^{\infty} e^{-\alpha' t} g(t) \, dt, \quad \text{where} \quad p = \alpha - i\omega
\]

= \[
L[g(t)]
\]

2.10. PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If the Fourier transforms of \( f(x) \) and \( g(x) \) are \( F(s) \) and \( G(s) \) respectively, then

(i) \( \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cdot G(s) \, ds = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx \)

(ii) \( \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx \)

where \( \bar{g}(x) \) stands for the complex conjugate.

Proof. (i) By definition of Fourier transform

\[
F(s) = F[f(x)] = \int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x} \, dx
\]

By inversion formula for Fourier transform

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cdot e^{i\omega x} \, ds
\]

and

\[
g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(s) \cdot e^{i\omega x} \, ds
\]

\[
\int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx = \int_{-\infty}^{\infty} f(x) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G(s)} \cdot \overline{e^{i\omega x}} \, ds \right] \, dx
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G(s)} \left[ \int_{-\infty}^{\infty} f(x) \overline{e^{i\omega x}} \, dx \right] \, ds
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G(s)} \cdot F(s) \, ds
\]

(Changing the order of integration)

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G(s)} \cdot F(s) \, ds
\]
(ii) By definition of Fourier transform
\[ F(s) = F[f(x)] = \int f(x) e^{i\omega x} \, dx \]

By inversion formula for Fourier transform
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} \, d\omega \]

Then
\[ F_s(s) = \frac{2}{4 + s^2}, \quad G_s(s) = \frac{3}{9 + s^2} \]
\[ F_s(s) = \frac{s}{4 + s^2}, \quad G_s(s) = \frac{s}{9 + s^2} \]

\[ \therefore f(x) = e^{-ax} \Rightarrow F_s(s) = \frac{a}{a^2 + s^2}, F_s(s) = \frac{s}{a^2 + s^2} \]

(i) Using Parseval's identity for Fourier cosine transforms, i.e.,
\[ \int_{0}^{\infty} F_s(s) G_s(s) \, ds = \int_{0}^{\infty} f(x) g(x) \, dx \]

We have
\[ \int_{0}^{\infty} \frac{2}{4 + s^2} \left( \frac{3}{9 + s^2} \right) ds = \int_{0}^{\infty} e^{-2x} \cdot e^{-3x} \, dx \]

\[ \Rightarrow \quad \frac{12}{\pi} \int_{0}^{\infty} \frac{ds}{(4 + s^2)(9 + s^2)} = \int_{0}^{\infty} e^{-5x} \, dx \]

\[ = \frac{e^{-5x}}{-5} \bigg|_{0}^{\infty} = -\frac{1}{5} (0 - 1) = \frac{1}{5} \]

\[ \therefore \int_{0}^{\infty} \frac{dt}{(4 + t^2)(9 + t^2)} = \frac{\pi}{60} \]

(ii) Using Parseval's identity for Fourier sine transforms, i.e.,
\[ \int_{0}^{\infty} F_s(s) G_s(s) \, ds = \int_{0}^{\infty} f(x) g(x) \, dx \]

We have
\[ \int_{0}^{\infty} \frac{s}{4 + s^2} \left( \frac{s}{9 + s^2} \right) ds = \int_{0}^{\infty} e^{-2x} \cdot e^{-3x} \, dx \]

\[ \Rightarrow \quad \frac{2}{\pi} \int_{0}^{\infty} \frac{s^2}{(4 + s^2)(9 + s^2)} ds = \int_{0}^{\infty} e^{-5x} \, dx \]

\[ = \frac{e^{-5x}}{-5} \bigg|_{0}^{\infty} = -\frac{1}{5} (0 - 1) = \frac{1}{5} \]

\[ \therefore \int_{0}^{\infty} \frac{dt}{(4 + t^2)(9 + t^2)} = \frac{\pi}{50} \]

(iii) Using Parseval's identity for Fourier cosine transforms, i.e.,
\[ \int_{0}^{\infty} F_s(s) G_s(s) \, ds = \int_{0}^{\infty} f(x) g(x) \, dx \]

We have
\[ \int_{0}^{\infty} \frac{s}{4 + s^2} \left( \frac{s}{9 + s^2} \right) ds = \int_{0}^{\infty} e^{-2x} \cdot e^{-3x} \, dx \]

\[ \Rightarrow \quad \frac{2}{\pi} \int_{0}^{\infty} \frac{s^2}{(4 + s^2)(9 + s^2)} ds = \int_{0}^{\infty} e^{-5x} \, dx \]

\[ = \frac{e^{-5x}}{-5} \bigg|_{0}^{\infty} = -\frac{1}{5} (0 - 1) = \frac{1}{5} \]

\[ \therefore \int_{0}^{\infty} \frac{dt}{(4 + t^2)(9 + t^2)} = \frac{\pi}{10} \]

(iii) Using Parseval's identity for Fourier sine transforms, i.e.,
\[ \int_{0}^{\infty} F_s(s) G_s(s) \, ds = \int_{0}^{\infty} f(x) g(x) \, dx \]

\[ \int_{0}^{\infty} \frac{dt}{(4 + t^2)(9 + t^2)} = \frac{\pi}{10} \]

\[ \Rightarrow \quad \int_{0}^{\infty} \frac{t^2}{(4 + t^2)(9 + t^2)} dt = \frac{\pi}{10} \]

Example 1. Using Parseval's identities, prove that

(i) \[ \int_{0}^{\infty} \frac{dt}{(4 + t^2)(9 + t^2)} = \frac{\pi}{60} \]

(ii) \[ \int_{0}^{\infty} \frac{t^2}{(4 + t^2)(9 + t^2)} dt = \frac{\pi}{10} \]

(iii) \[ \int_{0}^{\infty} \frac{dt}{(4 + t^2)(9 + t^2)} = \frac{\pi}{32} \]

(iv) \[ \int_{0}^{\infty} \frac{t^2}{(9 + t^2)^2} dt = \frac{\pi}{12} \]

Sol. Let \( f(x) = e^{2x} \) and \( g(x) = e^{-3x} \)
We have \[ \frac{2}{\pi} \int_0^\infty \frac{(2 e^{-2x})^2}{4 + s^2} \, ds = \int_0^\infty e^{-4x} \, dx \]

\[ = \frac{8}{\pi} \int_0^\infty \frac{ds}{(4 + s^2)^2} = \int_0^\infty e^{-4x} \, dx \]

\[ = \left[ \frac{e^{-4x}}{-4} \right]_0^\infty = -\frac{1}{4} (0 - 1) = \frac{1}{4} \]

\[ \Rightarrow \quad \int_0^\infty \frac{ds}{(4 + s^2)^2} = \frac{\pi}{32} \]

\[ \therefore \quad \int_0^\infty \frac{dt}{(4 + t^2)^2} = \frac{\pi}{32}. \]

(iv) Using Parseval’s identity for Fourier sine transform, i.e.,

\[ \frac{2}{\pi} \int_0^\infty [G_s(s)]^2 \, ds = \int_0^\infty [g(x)]^2 \, dx \]

We have \[ \frac{2}{\pi} \int_0^\infty \left( \frac{8}{9 + s^2} \right)^2 \, ds = \int_0^\infty e^{-2s^2} \, dx \]

\[ = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(9 + s^2)^2} \, ds = \int_0^\infty e^{-s} \, dx \]

\[ = \left[ \frac{e^{-s}}{-6} \right]_0^\infty = -\frac{1}{6} (0 - 1) = \frac{1}{6} \]

\[ \Rightarrow \quad \int_0^\infty \frac{s^2}{(9 + s^2)^2} \, ds = \frac{\pi}{12} \]

\[ \therefore \quad \int_0^\infty \frac{t^2}{(9 + t^2)^2} \, dt = \frac{\pi}{12}. \]

Example 2. Using Parseval’s identity, prove that

\[ \int_0^\infty \frac{\sin at}{t(a^2 + t^2)} \, dt = -\frac{\pi}{2a^2} (1 - e^{-a^2}). \]

Sol. Let \( f(x) = e^{-ax}, \ a > 0 \) and \( g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases} \)

then \( F_s(s) = \frac{a}{a^2 + s^2} \) and \( G_s(s) = \frac{\sin as}{s}. \)

Using Parseval’s identity for Fourier cosine transforms, i.e.,

\[ \frac{2}{\pi} \int_0^\infty F_s(s) G_s(s) \, ds = \int_0^\infty f(x) g(x) \, dx \]

We have \[ \frac{2}{\pi} \int_0^\infty \left( \frac{a}{a^2 + s^2} \right) \left( \frac{\sin as}{s} \right) \, ds = \int_0^\infty e^{-ax} \cdot g(x) \, dx \]

\[ \Rightarrow \quad \frac{2a}{\pi} \int_0^\infty \frac{\sin as}{s(a^2 + s^2)} \, ds = \int_0^\infty e^{-ax} \cdot g(x) \, dx + \int_0^a e^{-ax} \cdot g(x) \, dx \]

\[ = \int_0^\infty e^{-ax} \cdot 1 \, dx + \int_0^a e^{-ax} \cdot 0 \, dx \]

\[ = \left[ \frac{e^{-ax}}{-a} \right]_0^\infty + 0 = -\frac{1}{a} (e^{-a^2} - 1) = \frac{1}{a} (1 - e^{-a^2}) \]

\[ \Rightarrow \quad \int_0^\infty \frac{\sin as}{s(a^2 + s^2)} \, ds = \frac{\pi}{2a^2} (1 - e^{-a^2}) \]

\[ \therefore \quad \int_0^\infty \frac{\sin at}{t(a^2 + t^2)} \, dt = \frac{\pi}{2a^2} (1 - e^{-a^2}). \]

Example 3. Find the Fourier transform of

\[ f(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \]

and hence find the value of \( \int_0^\infty \frac{\sin t^2}{t^2} \, dt. \)

(M.U.D., Dec. 2010)

Sol. Fourier transform of \( f(x) \) is given by

\[ F(f(x)) = \int_0^\infty (1 - |x|) e^{iax} \, dx \]

\[ = \int_{-\infty}^0 (1 - |x|) e^{iax} \, dx + \int_0^1 (1 - |x|) e^{iax} \, dx + \int_1^\infty (1 - |x|) e^{iax} \, dx \]

\[ = \int_{-\infty}^0 dx \cdot [1 - (1 - |x|)] e^{iax} \, dx + \int_0^1 dx \cdot [1 - (1 - |x|)] e^{iax} \, dx + \int_1^\infty (1 - |x|) e^{iax} \, dx \]

\[ = \int_1^\infty (1 - |x|) \cos ax + i \sin ax \, dx \]

\[ = \int_1^\infty (1 - |x|) \cos ax \, dx + \int_1^\infty (1 - |x|) \sin ax \, dx \]

\[ = 2 \int_0^1 (1 - |x|) \cos ax \, dx \quad (\because \cos ax \text{ is an even function of } x \text{ and } (1 - |x|) \text{ is an odd function of } x) \]

\[ = - \frac{\sin ax}{a} \bigg|_0^1 \quad (\because \sin ax \text{ is an even function of } x \text{ where } x = 1 - |x|) \]

\[ = - \frac{\sin a}{a} \bigg|_0^1 \quad (\because |x| = x \text{ where } x > 0) \]

\[ = - 2 \int_0^1 \frac{\sin ax}{a} \, dx \]

\[ = 2 \left[ \left( 1 - \frac{\cos ax}{a} \right) \right]_0^1 \]

\[ = 2 \left[ 1 - \frac{\cos a}{a} \right] \]
Using Parseval's identity for Fourier transform.

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx \]

We have

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4(1-\cos s)}{s^5} \, ds = \int_{-1}^{1} (1-|x|)^2 \, dx \]

\[ \Rightarrow \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{(1-\cos s)}{s^5} \, ds = \int_{-1}^{1} (1-|x|)^2 \, dx \]

\[ \Rightarrow \frac{4}{\pi} \int_{0}^{\infty} \frac{(1-\cos s)}{s^5} \, ds = \int_{-1}^{1} (1-|x|)^2 \, dx \]

(\because Both integrands are even functions)

\[ \Rightarrow \frac{4}{\pi} \int_{0}^{\infty} \frac{(2 \sin^2 s/2)^2}{s^5} \, ds = \frac{1}{2} \int_{-1}^{1} (1-x^2) \, dx \]

\[ \Rightarrow \frac{16}{\pi} \int_{0}^{\infty} \frac{\sin^4 t}{t^5} \, dt = \frac{2}{3} \int_{-1}^{1} (1-x^2) \, dx \]

Putting \( \frac{s}{2} = t \), i.e., \( s = 2t \), we have

\[ \frac{16}{\pi} \int_{0}^{\infty} \frac{\sin^4 t}{t^5} \, dt = \frac{2}{3} \]

\[ \Rightarrow \int_{0}^{\infty} \frac{\sin^4 t}{t^5} \, dt = \frac{\pi}{3} \]

**EXERCISE 2.3**

1. Verify convolution theorem for \( f(x) = g(x) = e^{-x^2} \). (M.D.U. May 2011)

2. Using Parseval's identities, prove that

   (i) \( \int_{0}^{\infty} \frac{dx}{(x^2 + 1)^2} = \frac{\pi}{4} \)
   
   [Hint. Use Parseval's identity for Fourier sine and cosine transforms of \( f(x) = e^{-x} \)]

   (ii) \( \int_{0}^{\infty} \frac{x^2}{(x^2 + 1)^2} \, dx = \frac{\pi}{4} \)

   (M.D.U. 2005)

3. Using Parseval's identity, show that

   \( \int_{0}^{\infty} \frac{dt}{(a^2 + t^2)(b^2 + t^2)} = \frac{\pi}{2ab(a+b)} \)  

   [Hint. Use Parseval's identity for Fourier cosine transforms of \( f(x) = e^{-ax}, g(x) = e^{-bx} \)]

   (M.D.U. 2005)

4. If \( f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases} \) and \( F(s) = 2 \frac{\sin as}{s}, (s \neq 0) \), then prove that \( \int_{0}^{\infty} \frac{\sin^2 x}{x^2} \, dx = \frac{\pi}{2} \)

5. Using Parseval's identity, prove that

   (i) \( \int_{0}^{\infty} \frac{\sin^2 x}{x^2} \, dx = \frac{\pi}{2} \)
   
   (ii) \( \int_{0}^{\infty} \cos 2x \, dx = \pi \)

6. Using Parseval's identity, prove that

   (i) \( \int_{0}^{\infty} \frac{(1-\cos x)}{x} \, dx = \frac{\pi}{2} \)
   
   (ii) \( \int_{0}^{\infty} \frac{\sin^4 x}{x^2} \, dx = \frac{\pi}{4} \)

**2.11. FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION**

The Fourier transform of the function \( u(x, t) \) is given by

\[ F[u(x, t)] = \int_{-\infty}^{\infty} u e^{ix} \, dx \]

(i) Fourier transform of \( \frac{\partial u}{\partial x} \)

Suppose \( u \to 0 \) as \( x \to \pm \infty \), then the Fourier transform of \( \frac{\partial u}{\partial x} \) is given by

\[ F \left[ \frac{\partial u}{\partial x} \right] = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} \, e^{ix} \, dx = \int_{-\infty}^{\infty} e^{ix} \cdot \frac{\partial u}{\partial x} \, dx \]

(Integrating by parts)

\[ = \left[ e^{ix} \cdot u \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} ise^{ix} \cdot u \, dx \]

\[ = 0 - is \int_{-\infty}^{\infty} e^{ix} \cdot u \, dx \]

\[ \Rightarrow \quad = -is F(u) \]

Hence

\[ F \left[ \frac{\partial u}{\partial x} \right] = -is F(u) \]

(ii) Fourier transform of \( \frac{\partial^2 u}{\partial x^2} \)

Suppose \( u \) and \( \frac{\partial u}{\partial x} \to 0 \) as \( x \to \pm \infty \), then the Fourier transform of \( \frac{\partial^2 u}{\partial x^2} \) is given by

\[ F \left[ \frac{\partial^2 u}{\partial x^2} \right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} \, e^{ix} \, dx = \int_{-\infty}^{\infty} e^{ix} \cdot \frac{\partial^2 u}{\partial x^2} \, dx \]

(Appling general rule of integration by parts)

\[ = \left[ e^{ix} \cdot \frac{\partial u}{\partial x} - ise^{ix} \cdot u \right]_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} e^{ix} \cdot u \, dx \]
\[ \begin{align*}
&= 0 - \frac{3}{2} \int u \cdot e^{ax} \ dx \\
&= -s^3 F(u).
\end{align*} \]

Hence
\[ F \left[ \frac{\partial^4 u}{\partial x^4} \right] = -s^4 F(u). \]

(iii) Fourier transform of \( \frac{\partial^6 u}{\partial x^6} \)

Suppose \( u, \frac{\partial u}{\partial x}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^5 u}{\partial x^5} \to 0 \) as \( x \to \pm \infty \), then the Fourier transform of \( \frac{\partial^6 u}{\partial x^6} \) is given by

\[ F \left[ \frac{\partial^6 u}{\partial x^6} \right] = \int_0^\infty \frac{\partial^6 u}{\partial x^6} \cdot e^{i \omega x} \ dx \]

(Applied general rule of integration by parts)

\[ = e^{i \omega x} \cdot \frac{\partial^5 u}{\partial x^5} - i \omega e^{i \omega x} \frac{\partial^4 u}{\partial x^4} + (i \omega)^2 e^{i \omega x} \frac{\partial^3 u}{\partial x^3} - \cdots + (-i \omega)^6 u \]

\[ = 0 + (-i \omega)^6 \int_0^\infty u \cdot e^{i \omega x} \ dx = (-i \omega)^6 F(u) \]

Hence
\[ F \left[ \frac{\partial^6 u}{\partial x^6} \right] = (-i \omega)^6 F(u). \]

2.12. FOURIER SINE AND COSINE TRANSFORMS OF \( \frac{\partial^2 u}{\partial x^2} \)

The Fourier sine and cosine transforms of the function \( u(x, t) \) are given by

\[ F_s[u(x, t)] = \int_0^\infty u \sin \omega x \ dx \]
and
\[ F_c[u(x, t)] = \int_0^\infty u \cos \omega x \ dx. \]

(i) Fourier sine transform of \( \frac{\partial^2 u}{\partial x^2} \)

Suppose \( u \) and \( \frac{\partial u}{\partial x} \to 0 \) as \( x \to \infty \), then the Fourier transform of \( \frac{\partial^2 u}{\partial x^2} \) is given by

\[ F_s \left[ \frac{\partial^2 u}{\partial x^2} \right] = \int_0^\infty \frac{\partial u}{\partial x} \sin \omega x \ dx = \int_0^\infty \sin \omega x \frac{\partial u}{\partial x} \ dx \]

(Integrating by parts)

\[ = \left[ \sin \omega x \cdot \frac{\partial u}{\partial x} \right]_0^\infty - \int_0^\infty \cos \omega x \frac{\partial^2 u}{\partial x^2} \ dx \]

\[ = 0 - s \int_0^\infty \cos \omega x \frac{\partial u}{\partial x} \ dx \]

\[ = -s \left[ \cos \omega x \cdot u \right]_0^\infty + s \int_0^\infty u \sin \omega x \ dx \]

\[ = s(u)_{x=0} - s^2 F_s(u). \]

Hence
\[ F_s \left[ \frac{\partial^2 u}{\partial x^2} \right] = s(u)_{x=0} - s^2 F_s(u). \]

(ii) Fourier cosine transform of \( \frac{\partial^2 u}{\partial x^2} \)

Suppose \( u \) and \( \frac{\partial u}{\partial x} \to 0 \) as \( x \to \infty \), then the Fourier transform of \( \frac{\partial^2 u}{\partial x^2} \) is given by

\[ F_c \left[ \frac{\partial^2 u}{\partial x^2} \right] = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cdot \cos \omega x \ dx = \int_0^\infty \cos \omega x \frac{\partial^2 u}{\partial x^2} \ dx \]

(Integrating by parts)

\[ = \left[ \cos \omega x \cdot \frac{\partial u}{\partial x} \right]_0^\infty - \int_0^\infty -s \sin \omega x \cdot \frac{\partial u}{\partial x} \ dx \]

\[ = 0 - \left[ \sin \omega x \cdot u \right]_0^\infty + s \int_0^\infty u \cos \omega x \ dx \]

\[ = -s \left[ \sin \omega x \cdot u \right]_{x=0} - s \int_0^\infty u \cos \omega x \ dx \]

\[ = -s \left[ \sin \omega x \cdot u \right]_{x=0} - s^2 F_c(u) \]

\[ = -s \left[ \sin \omega x \cdot u \right]_{x=0} - s^2 F_c(u) \]

\[ = s \cdot \frac{\partial u}{\partial x} \]

\[ = F \left[ \frac{\partial u}{\partial x} \right]_{x=0} \]

\[ = F \left[ \frac{\partial u}{\partial x} \right]_{x=0} \]

\[ = F \left[ \frac{\partial u}{\partial x} \right]_{x=0} \]
Fourier transforms are very useful in solving boundary value problems. We take Fourier transform of the given partial differential equation using given boundary and initial conditions. The required solution is then obtained by taking corresponding inverse transform. The choice of particular transform to be employed depends on the boundary conditions of the problem.

(i) If the interval is $-\infty < x < \infty$ and if boundary conditions are

$$u \text{ and } \frac{du}{dx} \to 0 \text{ as } x \to \pm\infty$$

use infinite Fourier transform.

(ii) If the interval is $0 < x < \infty$ and

(a) boundary conditions are $u$ and $\frac{du}{dx} \to 0 \text{ as } x \to \infty$ and $u(x, t) = 0$ or $f(t)$ at $x = 0$ and $L$

all $t$, use Fourier sine transform.

(b) boundary conditions are $u$ and $\frac{du}{dx} \to 0 \text{ as } x \to \infty$ and $\frac{du}{dx} = 0$ or $f(t)$ at $x = 0$ and $L$

all $t$, use Fourier cosine transform.

For the interval $0 < x < \infty$, we always assume $u$ and $\frac{du}{dx} \to 0 \text{ as } x \to \infty$, even if it is not given in the problem.

(iii) If the interval is $0 < x < L$ and

(a) boundary conditions are $u(0, t) = u(L, t) = 0$ for all $t$, use finite Fourier sine transform.

(b) boundary conditions are $\frac{du}{dx}(0, t) = \frac{du}{dx}(L, t) = 0$ for all $t$, use finite Fourier cosine transform.

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**ILLUSTRATIVE EXAMPLES**

**Example 1.** Solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0$$

subject to the conditions

(i) $u = 0$, when $x = 0, \quad t > 0$

(ii) $u(x, t) = \begin{cases} 1, & 0 < x < L \\ 0, & x \geq L \end{cases}$, when $t = 0$

and (iii) $u(x, t)$ is bounded.

**Sol.** Given

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0$$

Boundary condition is $u(0, t) = 0$

---

**FOURIER TRANSFORMS**

Initial conditions are $u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$

and $u(x, t)$ is bounded.

Since $u(0, t)$ is given, we take Fourier sine transform of both sides of (1). Thus

$$F_s \left( \frac{\partial u}{\partial t} \right) = F_s \left( \frac{\partial^2 u}{\partial x^2} \right)$$

$$\Rightarrow \int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

Integrating by parts

$$\Rightarrow -s \int_0^\infty \sin sx \frac{du}{dx} \, dx = \left[ \sin sx \frac{du}{dx} \right]_0^\infty - \int_0^\infty \cos sx \frac{du}{dx} \, dx$$

$$= 0 - s \int_0^\infty \cos sx \frac{du}{dx} \, dx$$

$[\therefore \text{when } x \to \infty, \text{ } u \to 0 \text{ and when } x = 0, \text{ } u = u(0, t) = 0]$

$$\Rightarrow \int_0^\infty \frac{d}{dt} \int_0^\infty u \sin sx \, dx = -s^2 \int_0^\infty u \sin sx \, dx$$

$$\Rightarrow \frac{d\bar{u}}{dt} = -s^2 \bar{u}, \quad \bar{u}(s, t) = F_s[u(x, t)]$$

Separating the variables,

$$\frac{d\bar{u}}{\bar{u}} = -s^2 dt$$

Integrating

$$\log \bar{u} = -s^2 t + \log c$$

$$\Rightarrow \log \left( \frac{\bar{u}}{c} \right) = -s^2 t \quad \text{or} \quad \bar{u} = ce^{-s^2 t}$$

Putting $t = 0$ in (3),

$$c = \bar{u}(s, 0) = F_s[u(x, 0)] = \int_0^1 u(x, 0) \sin sx \, dx$$

$$= \int_0^1 u(x, 0) \sin sx \, dx + \int_1^\infty u(x, 0) \sin sx \, dx$$
\[ = \int_0^1 \sin sx \, dx + \int_1^\infty 0 \cdot \sin sx \, dx \]

\[ = \left[ \frac{\cos sx}{s} \right]_0^1 = \frac{1 - \cos s}{s} \]

\[ \therefore \text{From (3), } \tilde{u}_s(s, t) = \left[ \frac{1 - \cos s}{s} \right] e^{-st} \]

Taking its inverse Fourier sine transform, we get

\[ u(x, t) = \frac{2}{\pi} \int_0^\infty \left[ \frac{1 - \cos s}{s} \right] e^{-st} \sin sx \, ds \]

which is the required solution.

**Example 2.** Solve \( \frac{\partial V}{\partial t} = K \frac{\partial^2 V}{\partial x^2} \) for \( x > 0, t > 0 \) under the boundary conditions \( V = V_0 \)

when \( x = 0, t > 0 \) and the initial condition \( V = 0 \) when \( t = 0, x > 0 \).

**Sol.** Given

\[ \frac{\partial V}{\partial t} = K \frac{\partial^2 V}{\partial x^2}, \quad x > 0, t > 0 \]

Boundary condition is \( V(0, t) = V_0, t > 0 \)

Initial condition is \( V(x, 0) = 0, x > 0 \)

Since \( V(0, t) \) is given, we take Fourier sine transform of both sides of (1). Thus

\[ \mathcal{F}_s \left( \frac{\partial V}{\partial t} \right) = \mathcal{F}_s \left( K \frac{\partial^2 V}{\partial x^2} \right) \]

\[ \Rightarrow \int_0^\infty \frac{\partial V}{\partial t} \sin sx \, dx = K \int_0^\infty \frac{\partial^2 V}{\partial x^2} \sin sx \, dx \]

Integrating by parts

\[ \frac{d}{dt} \int_0^\infty V \sin sx \, dx = K \left[ \sin sx \cdot \frac{\partial V}{\partial x} \right]_0^\infty - \int_0^\infty \cos sx \cdot \frac{\partial V}{\partial x} \, dx \]

\[ = K \left[ 0 - \int_0^\infty \cos sx \cdot \frac{\partial V}{\partial x} \, dx \right] \]

\[ \Rightarrow \frac{d}{dt} \int_0^\infty V \sin sx \, dx = K s \cos x \cdot \frac{\partial V}{\partial x} \bigg|_0^\infty - \int_0^\infty \cos sx \cdot \frac{\partial V}{\partial x} \, dx \]

\[ = -Ks \left[ \cos x \cdot \frac{\partial V}{\partial x} \right]_0^\infty - \int_0^\infty \cos sx \cdot \frac{\partial V}{\partial x} \, dx \]

\[ = -Ks \left[ \cos x \cdot V \right]_0^\infty - \int_0^\infty \cos x \cdot \frac{\partial V}{\partial x} \, dx \]

\[ = -Ks \left[ -V_0 + \int_0^\infty V \sin sx \, dx \right] \]

\[ \therefore \text{when } x \to \infty, V \to 0 \text{ and when } x = 0, V = V(0, t) = V_0 \]

\[ \therefore \text{Taking its inverse Fourier sine transform, we get} \]

\[ V(x, t) = \frac{2V_0}{\pi} \int_0^\infty \frac{\sin sx}{s} \, ds - \int_0^\infty \frac{e^{-Kx/\pi t}}{s} \sin sx \, ds \]

\[ = \frac{2V_0}{\pi} \left[ \pi - \int_0^\infty \frac{e^{-Kx/\pi t}}{s} \sin sx \, ds \right] \]

\[ = \frac{2V_0}{\pi} \left[ \frac{\pi}{2} - \int_0^\infty \frac{e^{-Kx/\pi t}}{s} \sin sx \, ds \right] \]

\[ \therefore \text{V}(x, t) = V_0 \left[ 1 - \frac{2}{\pi} \int_0^\infty \frac{e^{-Kx/\pi t}}{s} \sin sx \, ds \right] \]

which is the required solution.
Example 3. The temperature \( u \) in a semi-infinite rod \( 0 \leq x < \infty \) is determined by the differential equation

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}
\]

subject to the conditions:

(i) \( u = 0 \) when \( t = 0, x \geq 0 \).

(ii) \( \frac{\partial u}{\partial x} = -\mu \) (a constant) when \( x = 0 \) and \( t > 0 \).

Determine the temperature formula.

Sol. Given \( \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \)

Boundary condition \( \frac{\partial u}{\partial x} = -\mu \) when \( x = 0, t > 0 \)

Initial condition \( u(x, 0) = 0 \)

Since \( \frac{\partial u}{\partial x} \) at \( x = 0 \) is given, we take Fourier cosine transform of both sides of (1). Thus

\[
F_c \left( \frac{\partial u}{\partial t} \right) = F_c \left( k \frac{\partial^2 u}{\partial x^2} \right)
\]

\[
\Rightarrow \int_0^\infty \frac{\partial u}{\partial t} \cos sx \, dx = k \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos sx \, dx \quad \text{Integrating by parts}
\]

\[
= \frac{d}{dt} \int_0^\infty u \cos sx \, dx = k \left[ \cos sx \cdot \frac{\partial u}{\partial x} \right]_0^\infty - \int_0^\infty -s \sin sx \cdot \frac{\partial u}{\partial x} \, dx
\]

\[
= k \left[ 0 - (-\mu) + s \int_0^\infty \sin sx \frac{\partial u}{\partial x} \, dx \right]
\]

\[
= k \left[ \mu + s \int_0^\infty \sin sx \, dx \right]
\]

\[
= k \left[ \mu + s \int_0^\infty \sin sx \, dx \right]
\]

\[
= k \left[ \mu + s \right] (1 - e^{-s^2})
\]

\[\therefore \frac{\partial u}{\partial x} \to 0 \text{ when } x \to \infty \quad \text{and} \quad \frac{\partial u}{\partial x} = -\mu \text{ when } x = 0\]

\[
\Rightarrow \int_0^\infty u \cos sx \, dx = k \int_0^\infty \cos sx \, dx = k \left[ \mu + s \right] \frac{s}{\pi}
\]

\[
\Rightarrow \frac{d}{dt} \int_0^\infty u \cos sx \, dx = k \mu - k s^2 \int_0^\infty u \cos sx \, dx
\]

\[
\Rightarrow \frac{d\tilde{u}}{dt} = k \mu - k s^2 \tilde{u}
\]

where \( \tilde{u} = \tilde{u}_c(s, t) = F_c [u(x, t)] \)

\[
\frac{d}{dt} e^{k s^2 t} \tilde{u} = k \mu
\]

which is a linear differential equation.

I.F. \( e^{k s^2 t} \)

\[
\therefore \text{ Its solution is}
\]

\[
\tilde{u}_c(s, t) = \int k \mu e^{k s^2 t} \, dt = \frac{k \mu}{k s^2} + c e^{-k s^2 t}
\]

or

\[
\tilde{u}_c(s, t) = \frac{\mu}{s^2} + c e^{-k s^2 t}
\]

Putting \( t = 0 \) in (3), we have

\[
\tilde{u}_c(s, 0) = \frac{\mu}{s^2} + c
\]

\[
\Rightarrow \quad c = \frac{\mu}{s^2} - \tilde{u}_c(s, 0) = \frac{\mu}{s^2} + F_c [u(x, 0)]
\]

\[
= \frac{\mu}{s^2} + \int_0^\infty 0 \cos sx \, dx
\]

\[
= \frac{\mu}{s^2}
\]

\[\therefore \text{From (3),} \quad \tilde{u}_c(s, t) = \frac{\mu}{s^2} \left( 1 - e^{-k s^2 t} \right)
\]

Taking its inverse Fourier cosine transform, we get

\[
u(x, t) = \frac{\mu}{\pi} \left( 1 - e^{-k s^2 t} \right) \cos sx \, ds
\]

\[
u(x, t) = \frac{2\mu}{\pi} \int_0^\infty \cos sx \left( 1 - e^{-k s^2 t} \right) \, ds
\]

which is the required solution.

Example 4. If the initial temperature of an infinite bar is given by

\[
\theta(x) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}
\]

Determine the temperature at any instant \( x \) and at any instant \( t \).

Sol. To determine the temperature \( \theta(x, t) \), we have to solve the one-dimensional heat-flow equation

\[
\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}, \quad t > 0
\]

subject to the initial condition \( \theta(x, 0) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \)
Taking Fourier transform of (1), we get
\[ \int \frac{\partial}{\partial t} e^{iax} \, dx = \mathcal{F} \left[ \frac{\partial}{\partial t} e^{iax} \right] \]
or
\[ \frac{d}{dt} \int \theta e^{iax} \, dx = c^2 \mathcal{F} \left[ -\theta \right] \]
or
\[ \frac{d\theta}{dt} = -c^2 \mathcal{F} \theta \] where \( \mathcal{F} \theta(t) = \Theta(0, t) F(\theta, x, t) \)

Now taking the Fourier transform of (2), we get
\[ \mathcal{F} \left[ \Theta(s, 0) e^{iax} \right] = \int_{-\infty}^{\infty} \Theta(x, 0) e^{iax} \, dx = \Theta_0 \left[ \mathcal{F} e^{iax} \right] \]
\[ = \Theta_0 \left[ \frac{e^{iax} - e^{-iax}}{ias} \right] = 2\Theta_0 \left[ \frac{e^{iax} - e^{-iax}}{2ias} \right] \]
\[ = 2\Theta_0 \frac{\sin sa}{s} \]

From (3), \( \frac{d\theta}{\theta} = -c^2 \mathcal{F} \theta \)

Integrating \( \log \theta = -c^2 \mathcal{F} \theta + \log A \) or \( \theta = Ae^{-c^2t} \)

Since \( \theta = \frac{2\Theta_0 \sin sa}{s} \) when \( t = 0 \), from (4), we get \( A = \frac{2\Theta_0 \sin sa}{s} \)

\[ \therefore \theta = \frac{2\Theta_0 \sin sa}{s} e^{-c^2t} \]

Taking its inverse Fourier transform, we get
\[ \Theta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\Theta_0 \sin sa}{s} e^{i\xi x} \cdot e^{-ic^2t} \, d\xi \]
\[ = \frac{\Theta_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cdot e^{-c^2t} \cos xs - i \sin xs \, d\xi \]
\[ = \frac{\Theta_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cdot e^{-c^2t} \cos xs \, d\xi - i \int_{-\infty}^{\infty} \frac{\sin as}{s} \cdot e^{-c^2t} \sin xs \, d\xi \]
\[ = \frac{\Theta_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cdot e^{-c^2t} \cos xs \, d\xi \]

(The second integral vanishes since its integrand is an odd function)

\[ = \frac{2\Theta_0}{\pi} \int_{0}^{\infty} \frac{\sin as}{s} \cdot e^{-c^2t} \cos xs \, ds \]
\[ = \frac{2\Theta_0}{\pi} \int_{0}^{\infty} \frac{\sin (a+x)s + \sin (a-x)s}{s} \, ds \]

which is the required solution.

Example 5. Use the method of Fourier transform to determine the displacement \( y(x, t) \) of an infinite string, given that the string is initially at rest and that the initial displacement is \( f(x) \), \( -\infty < x < \infty \).

Solving. The equation for the vibration of the string is
\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \]
subject to the initial conditions
\[ \left( \frac{\partial y}{\partial t} \right)_{t=0} = 0 \quad \text{and} \quad y(x, 0) = f(x) \]

Taking Fourier transform of (1), we get
\[ \frac{\partial^2 \tilde{y}}{\partial t^2} = c^2 \frac{\partial^2 \tilde{y}}{\partial x^2} \]
where \( \tilde{y} = F(y(x, t)) \)

or
\[ \frac{\partial^2 \tilde{y}}{\partial t^2} + c^2 \tilde{y} = 0 \]

Its solution is \( \tilde{y} = A \cos cxt + B \sin cxt \)

where \( A, B \) are constants.

Now taking Fourier transform of (2), we get
\[ \frac{\partial \tilde{y}}{\partial t} = 0 \quad \text{and} \quad \tilde{y} = F(s) \quad \text{when} \ t = 0 \]

Putting \( t = 0, \tilde{y} = F(s) \) in (3), we get \( A = F(s) \)

Also
\[ \frac{\partial \tilde{y}}{\partial t} = -cA \sin cxt + cB \cos cxt \]

Putting \( t = 0, \frac{\partial \tilde{y}}{\partial t} = 0 \), we get \( B = 0 \)

\[ \therefore \tilde{y} = F(s) \cos cxt \]

Taking inverse Fourier transform, we get
\[ y(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cos cxt \cdot e^{-isx} \, ds \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \frac{e^{-isx} + e^{isx}}{2} \cdot e^{-isx} \, ds \]
\[ = \frac{1}{4\pi} \left[ \int F(s) e^{-isx} \cdot e^{isx} + F(s) e^{isx} \cdot e^{isx} \cdot ds \right] \]
\[ = \frac{1}{2} \left[ f(x - ct) + f(x + ct) \right] \]

Example 6. Use the complex form of Fourier transform to show that
\[ u = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} f(u) e^{i\frac{(x-u)^2}{4u}} \, du \]
is the solution of the boundary value problem
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0; \quad u = f(x) \quad \text{when} \ t = 0. \]
Sol. Given \( \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \)  

Initial condition is \( u(x, 0) = f(x) \)

Taking Fourier transform of both sides of (1)

\[
\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{i\omega x} \, dx = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{i\omega x} \, dx \\
\frac{d}{dt} \int_{-\infty}^{\infty} u e^{i\omega x} \, dx = \left[ e^{i\omega x} \frac{\partial u}{\partial x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} i\omega e^{i\omega x} \frac{\partial u}{\partial x} \, dx \\
= 0 - i\omega \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} \, dx \\
= i\omega \left[ u \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} i\omega u \, dx \\
= -i\omega \left[ 0 - \int_{-\infty}^{\infty} u e^{i\omega x} \, dx \right] \\
\Rightarrow \frac{du}{dt} = -\omega^2 u
\]

Integrating

\[
\log u = -\omega^2 t + \log c \\
\Rightarrow u = ce^{-\omega^2 t}
\]

Putting \( t = 0, u(s, 0) = c \)

\( c = u(s, 0) = \int_{-\infty}^{\infty} u(x, 0) e^{i\omega x} \, dx \)  

\[
= \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx \\
[\text{from (2)}]
\]

∴ From (3),

\[
\bar{u} = e^{-\omega^2 t} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx \\
= e^{-\omega^2 t} \int_{-\infty}^{\infty} f(u) e^{i\omega u} \, du \\
[\text{by changing the variable } x \text{ to } u] \\
\]

Taking its inverse Fourier transform, we get

\[
u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ e^{-\omega^2 t} \int_{-\infty}^{\infty} f(u) e^{i\omega u} \, du \right] e^{i\omega x} \, d\omega \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2 t} f(u) e^{i\omega x} \, d\omega \, du \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2 t} f(u) \left[ \int_{-\infty}^{\infty} e^{i\omega (x-u)} \, du \right] \, d\omega \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[ \int_{-\infty}^{\infty} e^{-\frac{(x-u)^2}{2t}} \frac{e^{i\omega (x-u)}}{\sqrt{2\pi}} \, d\omega \right] \, du \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[ \int_{-\infty}^{\infty} e^{-\frac{(x-u)^2}{2t}} \frac{e^{i\omega (x-u)}}{\sqrt{2\pi}} \, d\omega \right] \, du \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{(x-u)^2}{2t}} \sqrt{2\pi} \, du \\
[\because \int e^{-y^2} \, dy = \sqrt{\pi} ] \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-\frac{(x-u)^2}{4t}} \, du.
Example 7. Using suitable transforms, solve the differential equation \( \frac{\partial^2 V}{\partial t^2} = \frac{1}{\partial x^2} \), \( 0 \leq x \leq \pi, \ t \geq 0 \) where \( V(0, t) = 0 = V(\pi, t) \) and \( V(x, 0) = V_0 \) constant.

Sol. Given \( \frac{\partial V}{\partial t} = \frac{1}{\partial x^2}, \ 0 \leq x \leq \pi, \ t \geq 0 \)

Boundary conditions are \( V(0, t) = 0 = V(\pi, t) \)

Initial condition is \( V(x, 0) = V_0 \)

Since the interval \( 0 \leq x \leq \pi \) is finite and \( V(0, t) \) and \( V(\pi, t) \) are given, we use finite Fourier sine transform.

Let \( \overline{V}_s(n, t) \) denote finite Fourier sine transform of \( V(x, t) \), then

\[
\overline{V}_s(n, t) = \int_0^\pi V(x, t) \sin \left( \frac{n\pi x}{\pi} \right) \, dx = \int_0^\pi V(x, t) \sin nx \, dx
\]

Taking the finite Fourier sine transform of both sides of (1), we have

\[
\int_0^\pi \frac{\partial V}{\partial t} \sin nx \, dx = \int_0^\pi \frac{\partial^2 V}{\partial x^2} \sin nx \, dx
\]

\[
\Rightarrow \frac{d}{dt} \int_0^\pi V \sin nx \, dx = \left[ \sin nx \frac{\partial V}{\partial x} \right]_0^\pi - \int_0^\pi \cos nx \frac{\partial V}{\partial x} \, dx
\]

\[
= 0 - n \int_0^\pi \cos nx \frac{\partial V}{\partial x} \, dx
\]

\[
= n \left[ \cos nx \cdot V \right]_0^\pi - n \int_0^\pi \sin nx \cdot V \, dx
\]

\[
= -n \left[ n + n \int_0^\pi V \sin nx \, dx \right]
\]

\[\therefore \ V = 0 \text{ at } x = 0 \text{ and } t \geq 0\]

which is the required solution.

Example 8. Using finite Fourier transform, find the solution of the wave equation

\( \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \) subject to the conditions:

\( u(0, t) = u(\pi, t) = 0, \ t > 0 \)

\( u(x, 0) = 3 \sin x + 4 \sin 4x \) and \( u_t(x, 0) = 0 \) for \( 0 < x < \pi \).

Sol. Given \( \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \ 0 \leq x \leq \pi, \ t > 0 \)

and \( u(0, t) = u(\pi, t) = 0, \ t > 0 \)

\( u(x, 0) = 3 \sin x + 4 \sin 4x \) for \( 0 < x < \pi \)

\( u_t(x, 0) = 0 \) for \( 0 < x < \pi \)
Taking finite Fourier sine transform of both sides of (1), we have

\[ \int_0^\infty \frac{\text{d}^2 u}{\text{d}x^2} \sin nx \, dx = a^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin nx \, dx \]

\[ = a^2 \left[ \sin nx \cdot \frac{\partial u}{\partial x} \right]_0^\infty - \int_0^\infty n \cos nx \cdot \frac{\partial u}{\partial x} \, dx \]

\[ = a^2 \left[ 0 - n \int_0^\infty \cos nx \cdot \frac{\partial u}{\partial x} \, dx \right] \]

\[ = -a^2n \left[ \cos nx \cdot u \right]_0^\infty - \int_0^\infty n \sin nx \cdot u \, dx \]

\[ = -a^2n \left[ 0 + n \int_0^\infty \sin nx \, dx \right] \quad \because \quad u(0, t) = u(\pi, t) = 0 \]

\[ = -a^2n^2 \int_0^\infty u \sin nx \, dx \]

\[ \Rightarrow \quad \frac{\text{d}^2 u}{\text{d}t^2} + a^2n^2 \sin n \pi = 0 \]

where

\[ \bar{u}_n = \bar{u}_n (n, t) = \int_0^\infty u(x, t) \sin \left( \frac{n\pi x}{\pi} \right) \, dx = \int_0^\infty u(x, t) \sin nx \, dx \]

A.E. is \( D^2 + a^2n^2 = 0 \) \( \Rightarrow D = \pm an \)

\[ \therefore \quad \bar{u}_n = c_1 \cos an + c_2 \sin an \]

Put \( t = 0 \) in (3), \( \bar{u}_n(n, 0) = c_1 \)

\[ \Rightarrow \quad c_1 = \bar{u}_n(n, 0) = \int_0^\infty u(x, 0) \sin nx \, dx \]

\[ = \int_0^\infty (3 \sin x + 4 \sin 4x) \sin nx \, dx \]

\[ = 3 \int_0^\infty \sin nx \sin x \, dx + 4 \int_0^\infty \sin nx \sin 4x \, dx \]

\[ = \frac{3}{2} \int_0^\infty 2 \sin nx \sin x \, dx + 2 \int_0^\infty 2 \sin nx \sin 4x \, dx \]

\[ = \frac{3}{2} \int_0^\infty [\cos (n - 1)x - \cos (n + 1)x] \, dx + 2 \int_0^\infty [\cos(n - 4)x - \cos (n + 4)x] \, dx \]

\[ = 0 \text{ except for } n = 1 \text{ and } n = 4 \]

For \( n = 1 \),

\[ c_1 = \bar{u}_n(n, 0) = \int_0^\infty u(x, 0) \sin x \, dx \]

\[ = \int_0^\infty (3 \sin x + 4 \sin 4x) \sin x \, dx \]

\[ = 3 \int_0^\infty \sin^2 x \, dx + 4 \int_0^\infty \sin 4x \cos x \, dx \]

\[ = \frac{3}{2} \int_0^\infty \cos 3x - \cos 5x \, dx + 4 \int_0^\infty \frac{1 - \cos 8x}{2} \, dx \]

\[ = \frac{3}{2} \left[ \frac{\sin 3x}{3} - \frac{\sin 5x}{5} \right]_0^\infty + 2 \left[ x - \frac{\sin 8x}{8} \right]_0^\infty = \frac{3\pi}{2} \]

For \( n = 4 \),

\[ c_1 = \bar{u}_n(n, 0) = \int_0^\infty u(x, 0) \sin 4x \, dx \]

\[ = \int_0^\infty (3 \sin x + 4 \sin 4x) \sin 4x \, dx \]

\[ = \frac{3}{2} \int_0^\infty \sin 4x \cos x \, dx + 4 \int_0^\infty \sin 2x \cos 4x \, dx \]

\[ = \frac{3}{2} \int_0^\infty \cos 3x - \cos 5x \, dx + 4 \int_0^\infty \frac{1 - \cos 8x}{2} \, dx \]

\[ = \frac{3}{2} \left[ \frac{\sin 3x}{3} - \frac{\sin 5x}{5} \right]_0^\infty + 2 \left[ x - \frac{\sin 8x}{8} \right]_0^\infty = \pi \]

\[ \therefore \quad c_1 = \begin{cases} \frac{3\pi}{2} & \text{for } n = 1 \\ \frac{2\pi}{2} & \text{for } n = 4 \end{cases} \]

Again from (3), \( \frac{\partial \bar{u}_n}{\partial t} = -c_1 \alpha \sin an + c_2 \alpha \cos an \)

Putting \( t = 0 \), \( \frac{\partial \bar{u}_n}{\partial t} = 0 \) \( \therefore \) when \( t = 0 \), \( \frac{\partial u}{\partial t} = 0 \) \( \therefore \) \( \frac{\partial u}{\partial t} = 0 \) for \( t = 0 \)

\[ 0 = c_2 \alpha \quad \Rightarrow \quad c_2 = 0 \]

\( \therefore \) (3) reduces to \( \bar{u}_n = c_1 \cos an \)

where \( c_1 = \frac{3\pi}{2} \) for \( n = 1 \) and \( c_1 = 2\pi \) for \( n = 4 \)
Taking inverse finite Fourier sine transform of (4), we have
\[ u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \tilde{u}_n(t, 0) \sin nx \]
\[ = \frac{2}{\pi} \left[ \frac{3 \pi}{2} \cos at \sin x + 2 \pi \cos 4 \theta \sin 4 \theta \right] \]
\[ = \frac{2}{\pi} \left[ \frac{2}{\pi} \cos at \sin x + 4 \cos 4 \theta \sin 4 \theta \right] \]
\[ \text{or} \]
which is the required solution.

**Example 9.** Using finite Fourier transform, solve \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \) subject to the conditions:

(a) \( u_x(0, t) = u_x(L, t) = 0, \quad 0 < x < L, \quad t > 0 \)

(b) \( u(x, 0) = x(6 - x), \quad 0 < x < L \).

**Sol.** Since the boundary conditions are \( u_x(0, t) = u_x(L, t) = 0 \), we take finite Fourier cosine transform.

Let \( \tilde{u}_x(n, t) \) denote finite Fourier cosine transform of \( u(x, t) \), then
\[ \tilde{u}_x(n, t) = \int_0^L u(x, t) \cos \left( \frac{n\pi x}{L} \right) \, dx \]

Taking finite Fourier cosine transform of both sides of the given equation, we get
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]

\[ \Rightarrow \int_0^L u \cos \left( \frac{n\pi x}{L} \right) \, dx = \frac{\partial}{\partial \theta} \left[ \int_0^L u \cos \left( \frac{n\pi x}{L} \right) \, dx \right] \]

\[ \Rightarrow \int_0^L u \cos \left( \frac{n\pi x}{L} \right) \, dx = \left[ \cos \left( \frac{n\pi x}{L} \right) \frac{\partial u}{\partial \theta} \right]_0^L - \frac{n\pi}{L} \int_0^L \sin \left( \frac{n\pi x}{L} \right) \frac{\partial u}{\partial \theta} \, dx \]

\[ = 0 + \frac{n\pi}{L} \sin \left( \frac{n\pi x}{L} \right) \frac{\partial u}{\partial \theta} \bigg|_0^L \]

\[ = \frac{n\pi}{L} \left[ \sin \left( \frac{n\pi x}{L} \right) \right]_0^L - \frac{n\pi}{6} \cos \left( \frac{n\pi x}{L} \right) \int_0^L u \, dx \]

\[ \Rightarrow \int_0^L u \cos \left( \frac{n\pi x}{L} \right) \, dx = \frac{n^2 \pi^2}{6} \int_0^L u \cos \left( \frac{n\pi x}{L} \right) \, dx \]

\[ \Rightarrow \frac{d}{dt} \int_0^L u \cos \left( \frac{n\pi x}{L} \right) \, dx = \frac{n^2 \pi^2}{6} \frac{d}{dt} \int_0^L u \cos \left( \frac{n\pi x}{L} \right) \, dx \]

\[ \Rightarrow \frac{d}{dt} \frac{d}{dt} \tilde{u}_x(n, t) = \frac{n^2 \pi^2}{6} \frac{d}{dt} \tilde{u}_x(n, t) \]

\[ \Rightarrow \frac{d}{dt} \tilde{u}_x(n, t) = -\frac{n^2 \pi^2}{6} \frac{d}{dt} \tilde{u}_x \]

\[ \Rightarrow \frac{d}{dt} \tilde{u}_x(n, t) = -\frac{n^2 \pi^2}{6} \frac{d}{dt} \tilde{u}_x \]

Integrating, \( \log \tilde{u}_x = -\frac{n^2 \pi^2}{36} t + \log A \quad \text{where} \quad A = A(n) \)

\[ \Rightarrow \tilde{u}_x(n, t) = A e^{-\frac{n^2 \pi^2}{36} t} \quad \Rightarrow \tilde{u}_x(n, t) = A(n)e^{-\frac{n^2 \pi^2}{36} t} \]

\[ \Rightarrow \tilde{u}_x(n, 0) = \frac{A(n)}{\pi n} \int_0^L x(6 - x) \cos \left( \frac{n\pi x}{L} \right) \, dx \]

\[ \Rightarrow \tilde{u}_x(n, 0) = \frac{A(n)}{\pi n} \int_0^L \left[ \frac{6}{n\pi} \cos \left( \frac{n\pi x}{L} \right) - \frac{6 \pi}{n} \sin \left( \frac{n\pi x}{L} \right) \right] \, dx \]

\[ \Rightarrow \tilde{u}_x(n, 0) = \frac{A(n)}{\pi n} \left[ \frac{6}{n\pi} \sin \left( \frac{n\pi x}{L} \right) - \frac{6 \pi}{n} \cos \left( \frac{n\pi x}{L} \right) \right]_0^L \]

\[ \Rightarrow \tilde{u}_x(n, 0) = \frac{A(n)}{\pi n} \left[ \frac{6}{n\pi} \sin \left( \frac{n\pi x}{L} \right) - \frac{6 \pi}{n} \cos \left( \frac{n\pi x}{L} \right) \right]_0^L \]

\[ \Rightarrow \tilde{u}_x(n, 0) = -\frac{216}{n^2 \pi^2} (1 - \cos \frac{n\pi}{L}) \]

\[ \Rightarrow \tilde{u}_x(n, 0) = -\frac{216}{n^2 \pi^2} (1 - \cos \frac{n\pi}{L}) e^{-\frac{n^2 \pi^2}{36} t} \]

\[ \Rightarrow \tilde{u}_x(n, 0) = \frac{216}{n^2 \pi^2} (1 + \cos \frac{n\pi}{L}) e^{-\frac{n^2 \pi^2}{36} t} \]

\[ \Rightarrow \tilde{u}_x(n, 0) = \frac{216}{n^2 \pi^2} (1 + \cos \frac{n\pi}{L}) e^{-\frac{n^2 \pi^2}{36} t} \]

To find inverse finite Fourier cosine transform, we need \( \tilde{u}_x(n, 0, t) \).

From (1), \( \tilde{u}_x(n, 0, t) = A(0) \)

From (2), \( A(0) = \tilde{u}_x(n, 0) \)

\[ = \int_0^L x(6 - x) \, dx = \left[ \frac{3x^2 - x^3}{3} \right]_0^L \]

\[ = 108 - 72 = 36 \]

\[ \Rightarrow \tilde{u}_x(n, 0, t) = 36 \]

Taking inverse finite Fourier cosine transform of (3), we have
\[ u(x, t) = \frac{1}{L} \tilde{u}_x(0, t) + \frac{2}{L} \sum_{n=1}^{\infty} \tilde{u}_x(n, t) \cos \frac{n\pi x}{L} \]

\[ = \frac{1}{6} \left[ \frac{2}{L} \sum_{n=1}^{\infty} \frac{216}{n^2 \pi^2} (1 + \cos \frac{n\pi}{L}) e^{-\frac{n^2 \pi^2}{36} t} \right] \]

\[ = \frac{1}{6} \left[ \frac{2}{L} \sum_{n=1}^{\infty} \frac{216}{n^2 \pi^2} (1 + \cos \frac{n\pi}{L}) e^{-\frac{n^2 \pi^2}{36} t} \right] \]

\[ = 6 - 72 \sum_{n=1}^{\infty} \frac{1 + \cos \frac{n\pi}{L}}{n^2} e^{-\frac{n^2 \pi^2}{36} t} \cos \frac{n\pi x}{L} \]

which is the required solution.
1. Solve the partial differential equation
\[ \frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0 \]
subject to the following conditions:
(a) \( u(0, t) = 0, \quad t \geq 0 \)
(b) \( u(x, 0) = e^{-x}, \quad x > 0 \)
(c) \( u \) and \( \frac{\partial u}{\partial x} \) tend to 0 as \( x \to \infty \).

2. Solve the equation \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0 \) where \( u(x, t) \) satisfies the conditions:
(a) \( \frac{\partial u}{\partial x} \bigg|_{x=0} = 0, \quad t > 0 \)
(b) \( u(x, 0) = \begin{cases} \sin x, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases} \)
(c) \( |u(x, t)| < M \).

3. The initial temperature along the length of an infinite bar is given by
\[ u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \]
If the temperature \( u(x, t) \) satisfies the equation \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \), find the temperature at any point of the bar at any time \( t \).
(M.D.U. Dec. 2005)

4. Determine the distribution of temperature in the semi-infinite medium \( x \geq 0 \), when the end \( x = 0 \) is maintained at zero temperature and the initial distribution of temperature is \( f(x) \).

5. (a) If the flow of heat is linear so that the variation of \( \theta \) (temperature) with \( x \) and \( y \) may be neglected and if it is assumed that no heat is generated in the medium, then solve the differential equation
\[ \frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} \]
by using Fourier transform, where \( -\infty < x < \infty \) and \( \theta = f(x) \)
when \( t = 0, f(x) \) being a function of \( x \).
(b) If the initial temperature of an infinite bar is given by
\[ u(x, 0) = \begin{cases} 1, & \text{for } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases} \]
determine the temperature of the infinite bar at any point \( x \) and at any time \( t > 0 \).
(U.P.T.U. 2005)

6. Solve the equation \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 \) subject to the conditions
(a) \( u(x, 0) = 1, \quad 0 < x < \pi \)
(b) \( u(0, t) = u(\pi, t) = 0, \quad t > 0 \)
using appropriate Fourier transform.

7. Use finite Fourier transform to solve \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \) given that
\[ u(0, t) = 0, \quad u(\pi, t) = 0, \quad u(x, 0) = 2x, \quad 0 < x < \pi, \quad t > 0. \]

8. Use finite Fourier transform to solve
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 4, \quad t > 0 \]
subject to the conditions:
(a) \( u(x, 0) = 2x, \quad 0 < x < 4 \)
(b) \( u(0, t) = u(4, t) = 0, \quad t > 0 \).

9. Solve \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 6, \quad t > 0 \) subject to the conditions:
(a) \( u(0, t) = u(6, t) = 0, \quad t > 0 \)
(b) \( u(x, 0) = \begin{cases} 1, & 0 < x < 3 \\ 0, & 3 < x < 6 \end{cases} \)

10. Solve \( \frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}, \) given
\[ (a) \ u(0, t) = u(n, t) = 0 \quad \text{for} \ t > 0 \]
\[ (b) \ u(x, 0) = \frac{1}{10} \sin x + \frac{1}{100} \sin 4x \]
\[ (c) \ u_x(x, 0) = 0 \quad \text{for} \ 0 < x < \pi. \]

11. Solve \( \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 6, \quad t > 0 \) subject to conditions \( u_x(0, t) = u_x(6, t) = 0, \quad u(x, 0) = 2x \).

12. Solve by using finite Fourier transform
\[ \frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 2 \]
subject to the conditions:
\[ u(0, t) = u(2, t) = 0, \quad u(x, 0) = x(2-x) \quad \text{and} \ u_x(x, 0) = 0. \]

Answers

1. \( u(x,t) = \frac{2}{\pi} \int_0^\pi \frac{s}{1 + s^2} e^{-s^2t} \sin xs \, ds \)
2. \( u(x,t) = \frac{2}{\pi} \int_0^\pi \left( \sin s + \cos s - 1 \right) e^{-s^2t} \cos xs \, ds \)
3. \( u(x,t) = \frac{2}{\pi} \int_0^\pi e^{-s^2t} \left( \sin (1+xs) + \sin (1-xs) \right) ds \)
4. \( u(x,t) = \frac{2}{\pi} \int_0^\pi f(s) e^{-s^2t} \sin sx \, ds \) where \( f(s) = F[f(x)] \)
5. (a) \( \theta(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(s) e^{-s^2t} e^{-isx} \, ds \) where \( \widehat{f}(s) = F[f(x)] \)
(b) \( \mu(x,t) = \int_{-\infty}^{\infty} \frac{2}{\pi} \sin^2 e^{-s^2t} e^{-isx} \, ds = \frac{1}{\pi} \int_0^\infty e^{-s^2t} \left| \sin (e+s) + \sin (e-s) \right| \, ds \)
6. \( u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{\cos \frac{n\pi}{a}} e^{-n^2t} \sin nx \) for \( x \in (a,b) \)
2.14. FOURIER TRANSFORM OF AN INTEGRAL

Theorem. Let \( f(t) \) be piecewise continuous on every interval \([- l, l]\) and \( \int_{-\infty}^{\infty} |f(t)| dt \) converge. Let \( F[f(t)] = F(s) \) and \( F(s) \) satisfies \( F(0) = 0 \). Then

\[
F\left[ \int_{-\infty}^{\infty} f(T) dT \right] = \frac{1}{i s} F(s).
\]

2.15. FOURIER TRANSFORM OF DIRAC-DELTA FUNCTION

Dirac delta function (or unit impulse function) \( \delta(t - a) \) is defined as

\[
\delta(t - a) = \lim_{k \to 0} \delta_k(t - a) \quad \text{where}
\]

\[
\begin{align*}
\delta_k(t - a) &= \begin{cases} 
0, & \text{for } t < a \\
1/k, & \text{for } a \leq t < a + k \\
0, & \text{for } t \geq a + k 
\end{cases} \\
F(\delta(t - a)) &= \int_{-\infty}^{\infty} \delta(t - a)e^{isT} dt \\
&= \lim_{k \to 0} \int_{a}^{a+k} \frac{1}{k} e^{isT} dt \\
&= \lim_{k \to 0} \frac{1}{k} \left[ \frac{e^{isT}}{is} \right]_{a}^{a+k} \\
&= \lim_{k \to 0} \frac{e^{isa + isk} - e^{isa}}{isk} \\
&= \lim_{k \to 0} e^{isa} \left( e^{isk} - 1 \right)/isk \\
&= e^{isa} \cdot 1 \\
&= e^{is\alpha}.
\end{align*}
\]
CHAPTER 3
Functions of a Complex Variable

3.1. INTRODUCTION

A complex number \( z \) is an ordered pair \((x, y)\) of real numbers and is written as
\[
z = x + iy, \quad \text{where } i = \sqrt{-1}.
\]

The real numbers \( x \) and \( y \) are called the real and imaginary parts of \( z \). In the Argand's diagram, the complex number \( z \) is represented by the point \( P(x, y) \). If \((r, \theta)\) are the polar coordinates of \( P \), then \( r = \sqrt{x^2 + y^2} \) is called the modulus of \( z \) and is denoted by \( |z| \). Also \( \theta = \tan^{-1} \frac{y}{x} \) is called the argument of \( z \) and is denoted by \( \arg z \). Every non-zero complex number \( z \) can be expressed as
\[
z = r (\cos \theta + i \sin \theta) = re^{i\theta}
\]

If \( z = x + iy \), then the complex number \( x - iy \) is called the conjugate of the complex number \( z \) and is denoted by \( \overline{z} \).

Clearly,
\[
|\overline{z}| = |z|, \quad |z|^2 = z \overline{z},
\]
\[
R(z) = \frac{z + \overline{z}}{2}, \quad I(z) = \frac{z - \overline{z}}{2i}.
\]

3.2. FUNCTION OF A COMPLEX VARIABLE

If \( x \) and \( y \) are real variables, then \( z = x + iy \) is called a complex variable. If corresponding to each value of a complex variable \( z = x + iy \) in a given region \( R \), there correspond one or more values of another complex variable \( w = u + iv \), then \( w \) is called a function of the complex variable \( z \) and is denoted by
\[
w = f(z) = u + iv
\]

For example, if \( w = z^2 \), where \( z = x + iy \) and \( w = f(z) = u + iv \)
then
\[
u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)
\]
\[
\Rightarrow \quad u = x^2 - y^2 \quad \text{and} \quad v = 2xy
\]
Thus \( u \) and \( v \), the real and imaginary parts of \( w \), are functions of the real variables \( x \) and \( y \).

\[
w = f(z) = u(x, y) + iv(x, y)
\]
3.3. EXponential Function Of A Complex Variable

Def. The exponential function of the complex variable \( z = x + iy \), where \( x \) and \( y \) are real, is defined as

\[
e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots
\]

where, \( e = 2.718 \) is the base of natural logarithms.

Replacing \( z \) by \( x + iy \), we have

\[
e^{x + iy} = 1 + \frac{(x + iy)^1}{1!} + \frac{(x + iy)^2}{2!} + \frac{(x + iy)^3}{3!} + \ldots
\]

Putting, \( x = 0 \), we get

\[
e^y = 1 + \frac{(iy)^1}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \ldots
\]

\[
e^y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots
\]

\[
e^{x + iy} = e^x \cos y + i e^x \sin y
\]

[Note. We have proved for all \( c \) and \( y \),

\[
e^{x + iy} = e^x \cos y + i e^x \sin y
\]

Changing \( i \) to \( -i \) (i.e., taking conjugates of both sides),

\[
e^{-y} = e^y \cos y - i e^y \sin y
\]

3.4. PERIODICITY

\( e^z \) IS A PERIODIC FUNCTION, WHERE \( z \) IS A COMPLEX VARIABLE

Proof. Let \( z = x + iy \)

then, by definition

\[
e^z = e^{x + iy} = e^x \cos y + i e^x \sin y
\]

\[
e^z = e^{x + iy + 2ni} = e^{x + iy}
\]

\[
e^z = e^{x + (y + 2n\pi)} = e^z + 2n\pi
\]

i.e., \( e^z \) remains unchanged when \( z \) is increased by any multiple of \( 2n\pi \).

\( \Rightarrow \) \( e^z \) is a periodic function with period \( 2n\pi \).

Example 1. Split up into real and imaginary parts:

(i) \( e^{5 + 3i} \)

(ii) \( e^{5 + 3i} \)

Sol. (i)

\[
e^{5 + 3i} = e^5 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)
\]

\[
e^{5 + 3i} = e^5 \cos \frac{\pi}{2} + i e^5 \sin \frac{\pi}{2}
\]

\[
e^{5 + 3i} = e^5 [0 + i \cdot 1] = ie^5
\]

\[
\Rightarrow \left[ \begin{array}{c}
\text{Re} \ e^{5 + 3i} = 0 \\
\text{Im} \ e^{5 + 3i} = e^5
\end{array} \right.
\]

(5 + 2i)^2 = 25 + 9i^2 + 30i = 16 + 30i

\[
\Rightarrow e^{(5 + 3i)^2} = e^{16 + 30i} = e^{16} \cos(30 + i \sin 30)
\]

\[
\Rightarrow \Re e^{(5 + 3i)^2} = e^{16} \cos 30, \quad \Im e^{(5 + 3i)^2} = e^{16} \sin 30
\]

(iii) \( e^{-6 \pi i} \)

\[
\Rightarrow \left[ \begin{array}{c}
e^{-6 \pi i} = e^{-6} \cos(\pi) + i e^{-6} \sin(\pi) \\
e^{-6 \pi i} = e^{-6} \cos \pi + i e^{-6} \sin \pi
\end{array} \right.
\]

\[
\Rightarrow \left[ \begin{array}{c}
e^{-6 \pi i} = e^{-6} \cos \pi \\
e^{-6 \pi i} = e^{-6} \cos \pi
\end{array} \right.
\]

Example 2. Find all values of \( z \) which satisfy

\( e^z = 1 + i \)

Sol. Since

\[
e^z = e^x \cos y + i e^x \sin y
\]

\[
e^z \cos y + i e^x \sin y = 1 + i
\]

Equating real parts \( e^x \cos y = 1 \)

Equating imaginary parts \( e^x \sin y = 1 \)

Squaring and adding (1) and (2), we get

\[
e^{2x} (\cos^2 y + \sin^2 y) = 1 + 1 \quad \text{or} \quad e^{2x} = 2
\]

\[
\Rightarrow 2x = \log 2, \quad x = \frac{1}{2} \log 2
\]

Dividing (2) by (1),

\[
\tan y = 1 = \tan \frac{\pi}{4}, \quad \text{where} \ n \text{ is an integer.}
\]

\[
\Rightarrow z = x + iy = \frac{1}{2} \log 2 + i \left[ n \pi + \frac{\pi}{4} \right], \quad \text{where} \ n = 0, \pm 1, \pm 2, \ldots...
\]

Example 3. Prove that

\[
\sin (\alpha - \theta) + e^{i\alpha} \sin \theta = \sin^2 \alpha \left( \sin (\alpha - \theta) + e^{i\alpha} \sin \theta \right)
\]

Sol. L.H.S. = \( [\sin \alpha \cos \theta - \sin \alpha \sin \theta] + [\cos \alpha - i \sin \alpha \sin \theta] \)

\[
= \sin \alpha \cos \theta - i \sin \alpha \sin \theta = [\sin \alpha \cos \theta + i \sin \alpha \sin \theta]
\]

R.H.S. = \( \sin^2 \alpha \left( \sin \alpha \cos \theta - i \sin \alpha \sin \theta \right) \)

\[
= \sin^2 \alpha \left( \sin \alpha \cos \theta - i \sin \alpha \sin \theta \right)
\]

\[
= \sin^2 \alpha \cdot \sin \alpha \cos \theta - i \sin^2 \alpha \sin \alpha \sin \theta
\]

L.H.S. = R.H.S.

Example 4. Given \( \frac{1}{P} = \frac{1}{L} + CP + \frac{1}{R} \) where \( L, P, R \) are real, express \( \rho \) in the form \( A e^{i\theta} \)

giving the values of \( A \) and \( \theta \).
\[ \frac{1}{\rho} = \frac{1}{R} + \frac{CP + \frac{1}{R}}{LPRi} \]

\[ \rho = \frac{LPRi}{(R - LP^2 CR) + LP^2} \]

\[ \frac{L^2 P^2 R + iLPR (R - LP^2 CR)}{(R - LP^2 CR)^2 + LP^2} = A \cos \theta + i \sin \theta, \text{ say} \]

Equating real and imaginary parts, we get

\[ A \cos \theta = \frac{L^2 P^2 R}{(R - LP^2 CR)^2 + LP^2} \quad \text{(1)} \]

\[ A \sin \theta = \frac{LPR (R - LP^2 CR)}{(R - LP^2 CR)^2 + LP^2} \quad \text{(2)} \]

Squaring and adding (1) and (2),

\[ A^2 = \frac{L^2 P^2 R^2 + L^2 P^2 R^2 (R - LP^2 CR)^2}{(R - LP^2 CR)^2 + LP^2} \]

\[ = \frac{L^2 P^2 R^2 [(R - LP^2 CR)^2 + LP^2]}{(R - LP^2 CR)^2 + LP^2} \]

\[ = \frac{L^2 P^2 R^2}{(R - LP^2 CR)^2 + LP^2} \]

\[ A = \frac{LPR}{\sqrt{(R - LP^2 CR)^2 + LP^2}} \]

Dividing (2) by (1),

\[ \tan \theta = \frac{\rho \cos \theta + i \sin \theta}{A} \]

\[ \rho = A \cos \theta + i \sin \theta, \text{ say} \]

\[ \theta = \tan^{-1} \left( \frac{R (1 - LP^2 CR)}{LPR} \right) \]

Hence

\[ \theta = \tan^{-1} \left( \frac{R (1 - LP^2 CR)}{LPR} \right) \]

where

\[ A = \frac{LPR}{\sqrt{(R - LP^2 CR)^2 + LP^2}} \]

\[ \tan \theta = \frac{\rho}{A} = \frac{1}{\rho} \]

\[ = \frac{1}{R} + \frac{CP + \frac{1}{R}}{LPRi} \]

\[ \Rightarrow \]

\[ \frac{1}{\rho} = \frac{1}{R} + \frac{CP + \frac{1}{R}}{LPRi} \]

\[ \Rightarrow \]

\[ \frac{LPRi}{(R - LP^2 CR) + LP^2} \]

\[ = \frac{(R - LP^2 CR) + LP^2}{(R - LP^2 CR) + LP^2} \]

\[ = \frac{L^2 P^2 R + iLPR (R - LP^2 CR)}{(R - LP^2 CR)^2 + LP^2} = A \cos \theta + i \sin \theta, \text{ say} \]

Equating real and imaginary parts, we get

\[ A \cos \theta = \frac{L^2 P^2 R}{(R - LP^2 CR)^2 + LP^2} \quad \text{(1)} \]

\[ A \sin \theta = \frac{LPR (R - LP^2 CR)}{(R - LP^2 CR)^2 + LP^2} \quad \text{(2)} \]

Squaring and adding (1) and (2),

\[ A^2 = \frac{L^2 P^2 R^2 + L^2 P^2 R^2 (R - LP^2 CR)^2}{(R - LP^2 CR)^2 + LP^2} \]

\[ = \frac{L^2 P^2 R^2 [(R - LP^2 CR)^2 + LP^2]}{(R - LP^2 CR)^2 + LP^2} \]

\[ = \frac{L^2 P^2 R^2}{(R - LP^2 CR)^2 + LP^2} \]

\[ A = \frac{LPR}{\sqrt{(R - LP^2 CR)^2 + LP^2}} \]

\[ \cos \theta = \frac{\rho \cos \theta + i \sin \theta}{A} \]

\[ \rho = A \cos \theta + i \sin \theta, \text{ say} \]

\[ \theta = \tan^{-1} \left( \frac{R (1 - LP^2 CR)}{LPR} \right) \]

Hence

\[ \rho = A \cos \theta + i \sin \theta, \text{ say} \]

where

\[ A = \frac{LPR}{\sqrt{(R - LP^2 CR)^2 + LP^2}} \]

\[ \theta = \tan^{-1} \left( \frac{R (1 - LP^2 CR)}{LPR} \right) \]

\[ 1 \quad 1 \]

\[ \rho = \frac{1}{R} + CP + \frac{1}{R} \]

\[ \Rightarrow \]

\[ \frac{LPRi}{(R - LP^2 CR) + LP^2} \]

\[ = \frac{(R - LP^2 CR) + LP^2}{(R - LP^2 CR) + LP^2} \]

\[ = \frac{L^2 P^2 R + iLPR (R - LP^2 CR)}{(R - LP^2 CR)^2 + LP^2} = A \cos \theta + i \sin \theta, \text{ say} \]

Equating real and imaginary parts, we get

\[ A \cos \theta = \frac{L^2 P^2 R}{(R - LP^2 CR)^2 + LP^2} \quad \text{(1)} \]

\[ A \sin \theta = \frac{LPR (R - LP^2 CR)}{(R - LP^2 CR)^2 + LP^2} \quad \text{(2)} \]

Squaring and adding (1) and (2),

\[ A^2 = \frac{L^2 P^2 R^2 + L^2 P^2 R^2 (R - LP^2 CR)^2}{(R - LP^2 CR)^2 + LP^2} \]

\[ = \frac{L^2 P^2 R^2 [(R - LP^2 CR)^2 + LP^2]}{(R - LP^2 CR)^2 + LP^2} \]

\[ = \frac{L^2 P^2 R^2}{(R - LP^2 CR)^2 + LP^2} \]

\[ A = \frac{LPR}{\sqrt{(R - LP^2 CR)^2 + LP^2}} \]

\[ \cos \theta = \frac{\rho \cos \theta + i \sin \theta}{A} \]

\[ \rho = A \cos \theta + i \sin \theta, \text{ say} \]

\[ \theta = \tan^{-1} \left( \frac{R (1 - LP^2 CR)}{LPR} \right) \]

Hence

\[ \rho = A \cos \theta + i \sin \theta, \text{ say} \]

where

\[ A = \frac{LPR}{\sqrt{(R - LP^2 CR)^2 + LP^2}} \]

\[ \theta = \tan^{-1} \left( \frac{R (1 - LP^2 CR)}{LPR} \right) \]

\[ 1 \quad 1 \]

\[ \rho = \frac{1}{R} + CP + \frac{1}{R} \]

\[ \Rightarrow \]

\[ \frac{LPRi}{(R - LP^2 CR) + LP^2} \]

\[ = \frac{(R - LP^2 CR) + LP^2}{(R - LP^2 CR) + LP^2} \]

\[ = \frac{L^2 P^2 R + iLPR (R - LP^2 CR)}{(R - LP^2 CR)^2 + LP^2} = A \cos \theta + i \sin \theta, \text{ say} \]

Equating real and imaginary parts, we get

\[ A \cos \theta = \frac{L^2 P^2 R}{(R - LP^2 CR)^2 + LP^2} \quad \text{(1)} \]

\[ A \sin \theta = \frac{LPR (R - LP^2 CR)}{(R - LP^2 CR)^2 + LP^2} \quad \text{(2)} \]

Squaring and adding (1) and (2),

\[ A^2 = \frac{L^2 P^2 R^2 + L^2 P^2 R^2 (R - LP^2 CR)^2}{(R - LP^2 CR)^2 + LP^2} \]

\[ = \frac{L^2 P^2 R^2 [(R - LP^2 CR)^2 + LP^2]}{(R - LP^2 CR)^2 + LP^2} \]

\[ = \frac{L^2 P^2 R^2}{(R - LP^2 CR)^2 + LP^2} \]

\[ A = \frac{LPR}{\sqrt{(R - LP^2 CR)^2 + LP^2}} \]

\[ \cos \theta = \frac{\rho \cos \theta + i \sin \theta}{A} \]

\[ \rho = A \cos \theta + i \sin \theta, \text{ say} \]

\[ \theta = \tan^{-1} \left( \frac{R (1 - LP^2 CR)}{LPR} \right) \]

Hence

\[ \rho = A \cos \theta + i \sin \theta, \text{ say} \]

where

\[ A = \frac{LPR}{\sqrt{(R - LP^2 CR)^2 + LP^2}} \]

\[ \theta = \tan^{-1} \left( \frac{R (1 - LP^2 CR)}{LPR} \right) \]
Multiplying the numerator and denominator by $e^{inz}

\frac{e^{iz} - e^{i2z}}{1 - e^{i2z}} = \frac{e^{iz} - e^{i2z}}{1 - e^{i2z}} = \tan z

\therefore e^{inz} = 1

\Rightarrow \tan z \text{ remains unchanged when } z \text{ is increased by any multiple of } \pi.

\because \tan z \text{ is a periodic function with period } \pi.

### 3.6. TRIGONOMETRIC IDENTITIES

If $z$ is a complex variable, prove that

(i) $\sin^2 z + \cos^2 z = 1$

(ii) $\sin 2z = 2 \sin z \cos z$

(iii) $\cos 2z = \cos^2 z - \sin^2 z = 2 \cos^2 z - 1 = 1 - 2 \sin^2 z$

(iv) $\tan 2z = \frac{2 \tan z}{1 - \tan^2 z}$

(v) $\sin \frac{-z}{3} = -\sin z$

(vi) $\sin 3z = 3 \sin z - 4 \sin^3 z$

(vii) $\tan 3z = \frac{3 \tan z - \tan^3 z}{1 - 3 \tan^2 z}$

#### Proof

(i) L.H.S. = $\sin^2 z + \cos^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2$

$= \frac{1}{4} (e^{2iz} + e^{-2iz} - 2) + \frac{1}{4} (e^{2iz} + e^{-2iz} + 2) = \frac{1}{2} + \frac{1}{2} = 1 = \text{R.H.S.}$

(ii) R.H.S. = $2 \sin z \cos z = 2 \cdot \frac{e^{iz} - e^{-iz}}{2i} \cdot \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{2iz} - e^{-2iz}}{2} = \sin 2z = \text{L.H.S.}$

(iii) $\cos^2 z - \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz} - e^{-iz}}{2}\right)^2$

$= \frac{1}{4} (e^{2iz} + e^{-2iz} + 2) - \frac{1}{4} (e^{2iz} - e^{-2iz} - 2) = \frac{e^{2iz} - e^{-2iz}}{2} = \cos 2z$  

$2 \cos^2 z - 1 = 2 \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - 1 = \frac{1}{2} (e^{2iz} + e^{-2iz} + 2) - 1 = \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$

$1 - 2 \sin^2 z = 1 - 2 \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = 1 + \frac{1}{2} (e^{2iz} - e^{-2iz} - 2) = \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$

Hence the result.

(iv) R.H.S. = $\frac{2 \tan z}{1 - \tan^2 z} = \frac{2 \cdot \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}}{1 - \left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right)^2}$

$= \frac{2 (e^{2iz} - e^{-2iz})}{i2 (e^{2iz} + e^{-2iz})} = \frac{e^{2iz} - e^{-2iz}}{i (e^{2iz} + e^{-2iz})} = \tan 2z = \text{L.H.S.}$

#### EXERCISE 3.1

1. If $z = x + iy$, find the real and imaginary parts of (i) $e^{z}$ (ii) $\exp(iz)$ (iii) $\frac{e^{z}}{1 - z^{2}e^{z}}$

2. Prove that

(i) $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

(ii) $\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

(iii) $\frac{\sin^2 z}{z^2} = 1 - \tan ^2 z$, where $z = e^{\alpha} i$

3. Prove that by a proper choice of $p$ and $q$, $pe^{2\alpha} + qe^{-2\alpha}$ can be made equal to $5 \cos 2\theta - 7 \sin 2\theta$.

4. If $z$ is a complex number, prove that

(i) $\cos (-z) = \cos z$

(ii) $\tan (-z) = -\tan z$

(iii) $\cos 3z = 4 \cos^3 z - 3 \cos z$

(iv) $\tan 3z = \frac{3 \tan z - \tan^3 z}{1 - 3 \tan^2 z}$
5. If \( z_1, z_2 \) are complex numbers, show that
\[
\begin{align*}
\sin (z_1 + z_2) &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \\
\frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2} &= \tan (z_1 + z_2), \\
\frac{\sin z_1 + \sin z_2}{\cos z_1 \cos z_2} &= \sin (z_1 + z_2), \\
\cos (z_1 - z_2) &= 2 \sin z_1 \sin z_2.
\end{align*}
\]

6. Show that
\[
\begin{align*}
\cos (\alpha + i\beta) &= \frac{1}{2} (e^{\alpha} + e^{i\beta}) \cos \alpha + \frac{1}{2} (e^{-\alpha} - e^{-i\beta}) \cos \alpha, \\
\sin (\alpha - i\beta) &= \frac{1}{2} (e^{\alpha} - e^{-i\beta}) \sin \alpha + \frac{i}{2} (e^{\alpha} + e^{-i\beta}) \cos \alpha.
\end{align*}
\]

7. If \( \alpha \) and \( \beta \) are the imaginary cube roots of unity, prove that
\[
c^{\alpha \beta} + e^{\beta \alpha} = \frac{1}{2} \left( \cos \frac{2\pi}{3} \pm \sin \frac{2\pi}{3} \right).
\]

Answers
\[
\begin{align*}
(i) & \quad e^{x+y} = e^x e^y, \\
(ii) & \quad \sin^2 x + \cos^2 y = 2 \cos 2xy, \\
(iii) & \quad \cos \theta - \lambda \cos (\theta - \phi) = \frac{\sin \theta - \lambda \sin (\theta - \phi)}{1 - 2\lambda \cos \theta + \lambda^2}.
\end{align*}
\]

3.7. LOGARITHMIC FUNCTION OF A COMPLEX VARIABLE

Definition. If \( \omega = e^z \), where \( z \) and \( \omega \) are complex numbers, then \( z \) is called a logarithm of \( \omega \) to the base \( e \). Thus \( \log \omega = z \).

1. Prove that \( \log \omega \) is a many-valued function.

Proof. We know that \( e^{2\pi n} = 1 \) and \( \sin 2\pi n = 0 \).

Let \( e^{\omega} = \omega \), then \( e^{\omega + 2\pi n} = e^{\omega} \).

By definition, \( \log \omega = z + 2\pi ni \), where \( n \) is zero, any +ve or -ve integer.

Thus if \( z \) be a logarithm of \( \omega \), so is \( z + 2\pi ni \).

Hence the logarithm of a complex number has infinite values and is thus a many-valued function.

Note. The value \( z + 2\pi ni \) is called the general value of \( \log \omega \) and is denoted by \( \log_0 \omega \).

Thus \( \log_0 \omega = z + 2\pi ni + \log_0 \omega \).

If \( \omega = x + iy \), then \( \log (x + iy) = 2\pi ni + \log (x + iy) \).

If we put \( n = 0 \), in the general value, we get the principal value of \( z \), i.e., \( \log_0 \omega \).

2. Prove that \( \log (-N) = \pi i + \log N \), where \( N \) is positive.

Proof. \( -N = N(-1) = N(\cos \pi + i \sin \pi) = N \cdot e^{\pi i} \). 

Hence \( \log (-N) = \log (N \cdot e^{\pi i}) = \log N + e^{\pi i} = \log N + \pi i \).

3. Separate \( \log (\alpha + i\beta) \) into real and imaginary parts.

Proof. Let \( \alpha + i\beta = r(\cos \theta + i \sin \theta) \) so that \( r = \sqrt{\alpha^2 + \beta^2}, \theta = \tan^{-1} \frac{\beta}{\alpha} \).

ILLUSTRATIVE EXAMPLES

Example 1. Prove that \( \log (1 + r e^{i\theta}) = \frac{1}{2} \log (1 + 2r \cos \theta + r^2) + i \tan^{-1} \frac{r \sin \theta}{1 + r \cos \theta} \).

Deduce that \( \log (1 + \cos \theta + i \sin \theta) = \log \left( \frac{2 \cos \frac{\theta}{2} + i \frac{\theta}{2}}{2} \right) \).

Sol. \( \log (1 + r e^{i\theta}) = \log \left( 1 + r(\cos \theta + i \sin \theta) \right) = \log \left( 1 + r \cos \theta + i \tan^{-1} \frac{r \sin \theta}{1 + r \cos \theta} \right) \).

\[= \frac{1}{2} \log \left( \left[ 1 + r \cos \theta \right]^2 + \left( r \sin \theta \right)^2 \right) + i \tan^{-1} \frac{r \sin \theta}{1 + r \cos \theta} \]

Now \( \log (1 + \cos \theta + i \sin \theta) = \log (1 + e^{i\theta}) \).

Putting \( r = 1 \) in (i),

\[\log (1 + \cos \theta + i \sin \theta) = \frac{1}{2} \log (1 + 2 \cos \theta + 1) + i \sin \theta \]

\[= \frac{1}{2} \log \left( \frac{2(1 + \cos \theta)}{2} \right) + i \sin \theta \]

\[= \frac{1}{2} \log \left( 2 \cos \frac{\theta}{2} \right) + i \tan^{-1} \left( \tan \frac{\theta}{2} \right) \]

\[= \frac{1}{2} \log \left( \frac{2 \cos \frac{\theta}{2}}{2} \right) + i \frac{\theta}{2} \]... (i)

Example 2. Find the general value of \( \log (-3) \).

Sol. \( -3 = 3(-1) = 3 \cos \pi = 3 e^{i\pi} \).

\[\log (-3) = \log (3 e^{i\pi}) = 2\pi ni + \log (3 e^{i\pi}) = 2\pi ni + 3 + i\pi = 3 + i(2n + 1)\pi \]
Example 3. Separate into real and imaginary parts Log \((4 + 3i)\). (M.D.U. Dec. 2006)

Sol. Let \(4 + 3i = r(\cos \theta + i \sin \theta)\)

Equating real and imaginary parts \(r \cos \theta = 4\), \(r \sin \theta = 3\)

Squaring and adding, \(r^2 = 16 + 9 = 25\) \(\therefore r = 5\)

Dividing, \(\tan \theta = \frac{3}{4}\)

\[\log (4 + 3i) = \log r + i \log \left(\cos \theta + i \sin \theta\right) = \log r + i \theta\]

\[= 2n \pi + \log r + i \log e^{i \theta} = 2n \pi + \log 5 + i \theta = \log 5 + 2ni + i \tan^{-1} \frac{3}{4}\]

\[\text{Re} \{\log (4 + 3i)\} = \log 5\]

\[\text{Im} \{\log (4 + 3i)\} = \left\{2n \pi + \tan^{-1} \frac{3}{4}\right\}\]

Example 4. Prove that \(\tan \left[i \log \frac{a - ib}{a + ib}\right] = \frac{2ab}{a^2 - b^2}\).

Sol. Let \(a + ib = r(\cos \theta + i \sin \theta)\)

Equating real and imaginary parts \(r \cos \theta = a\), \(r \sin \theta = b\)

Dividing, \(\tan \theta = \frac{b}{a}\)

Also \(a - ib = r(\cos \theta - i \sin \theta)\)

\[\text{L.H.S.} = \tan \left[i \log \frac{a - ib}{a + ib}\right] = \tan \left[i \log \frac{e^{-i \theta}}{e^{i \theta}}\right] = \tan \left[i (-2i \log e)\right] = \tan 2 \theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \frac{b}{a}}{1 - \left\{\frac{b}{a}\right\}^2} = \frac{2ab}{a^2 - b^2}\]

Example 5. Express \(\log \log i\) in the form \(A + iB\).

Sol.

\[i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = e^{i \frac{\pi}{2}}\]

\[\therefore \log i = 2n \pi i + \log e^{i \frac{\pi}{2}} = 2n \pi i + \frac{\pi}{2} = i(4n + 1) \frac{\pi}{2}\]

\[\therefore \log \log i = \log \left[i(4n + 1) \frac{\pi}{2}\right] = 2mn i + \log \left[i(4n + 1) \frac{\pi}{2}\right] = 2mn i + \log (4n + 1) \frac{\pi}{2} = 2mn i + \frac{\pi}{2} + \log (4n + 1) \frac{\pi}{2}\]

ILLUSTRATIVE EXAMPLES

Example 1. Prove that \(i^i\) is wholly real and find its principal value. Also show that the values of \(i^i\) form a G.P.

Sol. \(i^i = e^{\log i^i}\) [By definition]

\[= e^{i(2n \pi + \log 1)} = e^{i(2n \pi)} = e^{i(2n \pi + \log \cos \theta + i \sin \theta)/2}\]

\[= e^{i(2n \pi i \log e)} = e^{i(2n \pi i)(i \frac{\pi}{2})} = e^{i(4n + 1)i \frac{\pi}{2}} = e^{i(4n + 1)i \frac{\pi}{2}}\]

which is wholly real.

The principal value of \(i^i = e^{-\pi/2}\) (Putting \(n = 0\))

Putting \(n = 0, 1, 2, \ldots \ldots \) the values of \(i^i\) are \(e^{-\pi/2}, e^{3\pi/2}, e^{9\pi/2}, \ldots \ldots\)

which form a G.P. whose common ratio is \(e^{-\pi/2}\).

Example 2. If \(a + ib = \alpha + i \beta\), prove that \(a^2 + b^2 = e^{\alpha i + i \beta}\).

Sol. \(a + ib = \alpha + i \beta\)

\[\therefore a = \alpha\]

\[\therefore b = \beta\]

\[\therefore \alpha^2 + \beta^2 = \alpha^2 + \beta^2 = e^{\alpha i + i \beta}\]

(K.U.K. 2005)
EQUATING REAL AND IMAGINARY PARTS
\[ \alpha = e^{i \alpha \theta} \cdot \cos \left( \frac{4n + 1}{2} \right); \quad \beta = e^{i \beta \theta} \cdot \sin \left( \frac{4n + 1}{2} \right) \]
Squaring and adding,
\[ \alpha^2 + \beta^2 = e^{i(4n + 1)\theta} \cos^2 \left( \frac{4n + 1}{2} \right) + \sin^2 \left( \frac{4n + 1}{2} \right) \]

Example 3. Considering only the principal value, prove that the real part of
\[ (1 + i \sqrt{3})^{1 + i \sqrt{3}} \text{ is } 2e^{\frac{i \sqrt{3}}{3} \log 2} \]

Sol. \( (1 + i \sqrt{3})^{1 + i \sqrt{3}} = e^{(1 + i \sqrt{3}) \log (1 + i \sqrt{3})} \)
\[ = e^{(1 + i \sqrt{3}) \left[ \frac{1}{2} \log(1 + i \sqrt{3}) + i \tan^{-1}(1 \sqrt{3}) \right]} \]
\[ = e^{(1 + i \sqrt{3}) \left[ \frac{1}{2} \log 2 + i \frac{\pi}{3} \right]} \]
\[ = e^{\frac{1}{2} \log 2} \cdot e^{\frac{i \pi}{3}} \left[ \cos \left( \frac{\pi}{3} + \frac{1 \sqrt{3} \log 2}{2} \right) + i \sin \left( \frac{\pi}{3} + \frac{1 \sqrt{3} \log 2}{2} \right) \right] \]
\[ = 2e^{\frac{i \pi}{2}} \left[ \cos \left( \frac{\pi}{3} + \frac{1 \sqrt{3} \log 2}{2} \right) + i \sin \left( \frac{\pi}{3} + \frac{1 \sqrt{3} \log 2}{2} \right) \right] \]
\[ \Rightarrow \text{Real part of } (1 + i \sqrt{3})^{1 + i \sqrt{3}} = 2e^{\frac{i \sqrt{3}}{3} \log 2} \]

Example 4. If \( e^{-i \omega \theta} = A + iB \) and only principal values are considered, prove that
(a) \( \tan \frac{\pi A}{B} \]
(b) \( A^2 + B^2 = e^{-i \beta \theta} \)

Sol. \( e^{-i \omega \theta} = A + iB \Rightarrow A^2 + B^2 = e^{i \omega \theta} \cdot (A + iB) \cdot (A + iB) \)
\[ = e^{i \omega \theta} \cdot (A^2 - B^2 + 2iAB) \quad \text{Taking principal values only} \]
\[ = e^{i \omega \theta} \cdot (A^2 + B^2) \]
\[ = e^{i \beta \theta} \cdot A^2 + B^2 \]

Equating real and imaginary parts
\[ A = e^{i \beta \theta} \cos \frac{\pi A}{2} \]
\[ B = e^{i \beta \theta} \sin \frac{\pi A}{2} \]

Dividing (ii) by (i), \( \tan \frac{\pi A}{2} = \frac{B}{A} \)

Squaring and adding (i) and (ii), \( A^2 + B^2 = e^{-i \beta \theta} \left( \cos^2 \frac{\pi A}{2} + \sin^2 \frac{\pi A}{2} \right) = e^{-i \beta \theta} \)

Example 5. If \( (a + ib)^n = m + n \), then prove that \( \frac{y}{x} = \tan^{-1} \left( \frac{b}{a} \right) \) when only principal values are considered.

Sol. \( (a + ib)^n = m + n \)
Taking log of both sides, \( \log(a + ib)^n = \log(m + n) \)

or \( p \log(a + ib) = (x + iy) \log m \)

or \[ p \left[ \frac{1}{2} \log(a^2 + b^2) + i \tan^{-1} \frac{b}{a} \right] = x \log m + iy \log m \]

(Considering only the principal values)

Equating real and imaginary parts \( x \log m = \frac{1}{2} p \log(a^2 + b^2) \) \( \quad \cdots (i) \)
\[ y \log m = p \tan^{-1} \frac{b}{a} \]
\[ \quad \cdots (ii) \]

Dividing (ii) by (i), \( \frac{y}{x} = \frac{p \tan^{-1} \frac{b}{a}}{\frac{1}{2} p \log(a^2 + b^2)} = 2 \tan^{-1} \frac{b}{a} \)

Example 6. If \( \tan \log(x + iy) = a + ib \) and \( a^2 + b^2 \neq 1 \), then prove that
\[ \tan \log(x^2 + y^2) = \frac{2a}{1 - a^2 - b^2} \]

Sol. \( \tan \log(x + iy) = a + ib \) \( \quad \cdots (i) \)
\[ \Rightarrow \tan \log(x - iy) = a - ib \]
\[ \Rightarrow \tan \log(x^2 + y^2) = \tan \log(x + iy)(x - iy) = \frac{a + ib + a - ib}{1 - (a + ib)(a - ib)} = \frac{2a}{1 - a^2 - b^2} \]

\( \Rightarrow \) where \( a^2 + b^2 \neq 1 \).

EXERCISE 3.2
1. Find the general value of
   (i) \( \log(-i) \)
   (ii) \( \log(1 + i) \)
2. Prove that
   (i) \( \log \left( \frac{x - i}{x + i} \right) = \pi - 2 \tan^{-1} x \)
   (ii) \( \cos \left( \log \left( \frac{a + ib}{a - ib} \right) \right) = \frac{a^2 - b^2}{a^2 + b^2} \)
   (iii) \( \beta = e^{-i(\pi + 1)x} \)
   (iv) \( \log i^3 = \left( \frac{-2 \pi + 1}{2} \right) \pi \)
3. Show that
   (i) \( \log(1 + i \tan \alpha) = \log \sec \alpha + i \alpha \)
   (ii) \( \log \left( \frac{3 - i}{3 + i} \right) = 2 \left( \pi - \tan^{-1} \frac{1}{3} \right) \)
FUNCTIONS OF A COMPLEX VARIABLE

3.9. HYPERBOLIC FUNCTIONS

1. Definitions. For all values of \( z \), real or complex

(i) the quantity \( e^z - e^{-z} \) is called hyperbolic sine of \( z \) and is written as \( \sinh z \)

(ii) the quantity \( e^z + e^{-z} \) is called hyperbolic cosine of \( z \) and is written as \( \cosh z \).

Thus \( \sinh x = \frac{e^x - e^{-x}}{2} \); \( \cosh x = \frac{e^x + e^{-x}}{2} \).

The other hyperbolic functions are defined in terms of hyperbolic sine and cosine as follows:

\[
\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \quad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}};
\]

\[
\text{sech} x = \frac{2}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad \text{cosech} x = \frac{2}{\sinh x} = \frac{2}{e^x - e^{-x}}.
\]

Note. \( \sinh 0 = \frac{e^0 - e^{-0}}{2} = 0 \); \( \cosh 0 = \frac{e^0 + e^{-0}}{2} = 1 \);

\[
\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x; \quad \cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x}.
\]

2. (a) Relations between hyperbolic and trigonometric functions

\[
\cos \theta = \frac{e^\theta + e^{-\theta}}{2}; \quad \sin \theta = \frac{e^\theta - e^{-\theta}}{2i}.
\]

Putting \( \theta = ix \) in these equations, we get

\[
\cos (ix) = \frac{e^{i(x)} + e^{-i(x)}}{2} = \frac{e^x + e^{-x}}{2}; \quad \sin (ix) = \frac{e^{i(x)} - e^{-i(x)}}{2i} = \frac{e^x - e^{-x}}{2i} = \frac{i^2(e^x - e^{-x})}{2i} = \frac{e^x - e^{-x}}{2i} = i \sinh x
\]

\[
\tan (ix) = \frac{\sin (ix)}{\cos (ix)} = \frac{i \sinh x}{\cosh x} = i \tanh x; \quad \cot (ix) = \frac{\cos (ix)}{\sin (ix)} = \frac{i \cosh x}{i \sinh x} = \frac{\cosh x}{i \sinh x} = -i \cot x;
\]

\[
\sec (ix) = \frac{1}{\cos (ix)} = \frac{1}{\cosh x} = \text{sech} x; \quad \cosec (ix) = \frac{1}{\sin (ix)} = \frac{1}{i \sinh x} = \frac{1}{i^2 \sinh x} = -i \cosech x.
\]

(b) By Definition, \( \sinh \theta = \frac{e^\theta - e^{-\theta}}{2}; \quad \cosh \theta = \frac{e^\theta + e^{-\theta}}{2}; \quad \tanh \theta = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}} \)
Thus tanh \( x \) remains unchanged when \( x \) is increased by any multiple of \( \pi i \). Hence \( \text{tanh } x \) is a periodic function and its period is \( \pi i \).

Note: cosech \( x \), sech \( x \) and coth \( x \) being reciprocals of sinh \( x \), cosh \( x \) and tanh \( x \) respectively, are also periodic functions with periods \( 2\pi i \), \( 2\pi i \) and \( \pi i \) respectively.

3.10. FORMULAE OF HYPERBOLIC FUNCTIONS

1. Prove that (a) \( \cosh^2 x - \sinh^2 x = 1 \), (b) \( \sech^2 x + \tanh^2 x = 1 \), (c) \( \coth^2 x - \cosech^2 x = 1 \)

Proof. (a) For all values of \( \theta \), \( \cos^2 \theta + \sin^2 \theta = 1 \)

Putting \( \theta = ix \), we get \( \cosh^2 (ix) + \sinh^2 (ix) = 1 \) or \( \cosh^2 x + i \sinh x = 1 \) [\( \because \cosh x = \cosh x; \sinh (ix) = i \sin x \)]

or \( \cosh^2 x - \sinh^2 x = 1 \) [\( \because i^2 = -1 \)]

(b) We know that \( \cosh^2 x - \sinh^2 x = 1 \)

Dividing both sides by \( \cosh^2 x \), we have

\[ 1 - \tanh^2 x = \sec^2 x \Rightarrow \sec^2 x + \tanh^2 x = 1 \]

(c) We know that \( \cosh^2 x = \sinh^2 x = 1 \)

Dividing both sides by \( \sinh^2 x \), we have

\[ \cosh^2 x - \cosech^2 x = \cosh^2 x - \cosech^2 x = 1 \]

2. Prove that (a) \( \sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \)
(b) \( \cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \)
(c) \( \tanh (x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \)

Proof. (a) \( \sinh (x \pm y) = \lim_{t \to 0} \frac{\sinh(x + ty) - \sinh(x - ty)}{2t} \)

\[ = \frac{1}{i} \int \left( e^{ix} \cos ty + \sin ty \right) dt \]

\[ = \frac{1}{i} \left( \sinh x \cosh y \pm \cosh x \sinh y \right) \]

(b) \( \cosh (x \pm y) = \lim_{t \to 0} \frac{e^{(x + ty)} - e^{-(x - ty)}}{2t} \)

\[ = \frac{1}{i} \int \left( e^{ix} \cos ty - \sin ty \right) dt \]

\[ = \cosh x \cosh y \pm \sinh x \sinh y \]

(c) \( \tanh (x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \)

Dividing the numerator and denominator by \( \cosh x \cosh y = \tanh x \pm \tanh y \)

\[ = 1 \pm \tanh x \tanh y \]

3. Prove that (a) \( \sinh 2x = 2 \sinh x \cosh x = \frac{2 \tanh x}{1 - \tanh^2 x} \)
(b) \( \cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x} \)

(c) \( \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \)

**Proof.** (a) We know that \( \sin 2\theta = 2 \sin \theta \cos \theta \)
Putting \( \theta = ix \), we get \( \sin (2ix) = 2 \sin (ix) \cos (ix) \) or \( i \sinh 2x = 2 . i \sinh x \cosh x \)

\[ \sinh 2x = 2 \sinh x \cosh x \]

Also \( \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \)
Putting \( \theta = ix \), we get \( \sin (2ix) = \frac{2 \tan x}{1 + \tan^2 x} \) or \( \sinh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \)

or \( \sinh 2x = \frac{2 \tanh x}{1 - \tanh^2 x} \)

(b) We know that \( \cosh 2\theta = \cosh^2 \theta - \sinh^2 \theta \)
Putting \( \theta = ix \), we get \( \cos (2ix) = \cos^2 (ix) - \sin^2 (ix) \) or \( \cosh 2x = (\cosh x)^2 - (i \sin x)^2 \)

\[ \cosh 2x = \cosh^2 x + \sinh^2 x \]

We know that \( \cosh 2\theta = 2 \cosh^2 \theta - 1 \)
Putting \( \theta = ix \), we get \( \cos (2ix) = 2 \cos^2 (ix) - 1 \) or \( \cosh 2x = 2 \cosh^2 x - 1 \)

**Cor.**
\[ \cosh^2 x = \frac{\cosh 2x + 1}{2} \]

We know that \( \cosh 2\theta = 1 - 2 \sin^2 \theta \)
Putting \( \theta = ix \), we get \( \cos (2ix) = 1 - 2 \sin^2 (ix) \)

or \( \cosh 2x = 1 - 2 (i \sinh x)^2 = 1 + 2 \sinh^2 x \)

**Cor.**
\[ \sinh^2 x = \frac{\cosh 2x - 1}{2} \]

We know that \( \cosh 2\theta = 1 - \tan^2 \theta \)
Putting \( \theta = ix \), we get \( \cos (2ix) = 1 - \tan^2 (ix) \)

or \( \cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh x} \)

(c) We know that \( \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \)
Putting \( \theta = ix \), we get \( \tan (2ix) = \frac{2 \tan (ix)}{1 - \tan^2 (ix)} \)

or \( i \tanh 2x = \frac{2 \tanh x}{1 - (i \tanh x)^2} \)

or \( \tanh 2x = \frac{2 \tanh x}{1 - \tanh^2 x} \)

\[ \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \]

**4. Prove that** (a) \( \sinh 3x = 3 \sinh x + 4 \sinh^3 x \)
(b) \( \cosh 3x = 4 \cosh^3 x - 3 \cosh x \)
(c) \( \tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x} \)

**Proof.** (a) We know that \( \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta \)
Putting \( \theta = ix \), we get \( \sin (3ix) = 3 \sin (ix) - 4 \sin^3 (ix) \)

or \( i \sinh 3x = 3i \sinh x - 4 (i \sinh x)^3 \)

or \( i \sinh 3x = 3i \sinh x + 4 i \sinh^3 x \)

or \( \sinh 3x = 3 \sinh x + 4 \sinh^3 x \)

(b) We know that \( \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \)
Putting \( \theta = ix \), we get \( \cos (3ix) = 4 \cos^3 (ix) - 3 \cos (ix) \)

or \( \cosh 3x = 4 \cosh^3 x - 3 \cosh x \)

(c) We know that \( \tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \)
Putting \( \theta = ix \), we get \( \tan (3ix) = \frac{3 \tan (ix) - \tan^3 (ix)}{1 - 3 \tan^2 (ix)} \)

or \( i \tanh 3x = \frac{3 i \tanh x - (i \tanh x)^3}{1 - 3 (i \tanh x)^2} \)

or \( i \tanh 3x = \frac{3 i \tanh x + i \tanh^3 x}{1 + 3 \tanh^2 x} \)

or \( \tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x} \)

**5. Prove that**

(i) \( 2 \sinh A \cosh B = \sinh (A + B) + \sinh (A - B) \)
(ii) \( 2 \cosh A \sinh B = \sinh (A + B) - \sinh (A - B) \)
(iii) \( 2 \cosh A \cosh B = \cosh (A + B) + \cosh (A - B) \)
(iv) \( 2 \sinh A \sinh B = \cosh (A - B) - \cosh (A + B) \)

**Proof.** We shall prove only the last result.
The first three are left as an exercise for the student.
We know that \( 2 \sin x \sin y = \cos (x - y) - \cos (x + y) \)
Putting \( x = iA; \ y = iB \), we get \( 2 \sin (iA) \sin (iB) = \cos (A - B) - \cos (A + B) \)

or \( 2 \cdot i \sinh A \cdot i \sinh B = \cosh (A - B) - \cosh (A + B) \)

or \( -2 \sinh A \sinh B = \cosh (A - B) - \cosh (A + B) \)

[:: \( i^2 = -1 \)]

**6. Prove that**

(i) \( \sinh C + \sinh D = 2 \sinh \frac{C + D}{2} \cosh \frac{C - D}{2} \)
(ii) \( \sinh C - \sinh D = 2 \cosh \frac{C + D}{2} \sinh \frac{C - D}{2} \)
(iii) \( \cosh C + \cosh D = 2 \cosh \frac{C + D}{2} \cosh \frac{C - D}{2} \)

(iv) \( \cosh C - \cosh D = 2 \sinh \frac{C - D}{2} \sinh \frac{C + D}{2} \)

**Proof.** We shall prove only the last result. The first three are left as an exercise for the student.

We know that \( \cos x - \cos y = 2 \sin \frac{x + y}{2} \sin \frac{y - x}{2} \)

Putting \( x = iA \) and \( y = iB \), we get

\[
\cos (iA) - \cos (iB) = 2 \sin \left( \frac{A + B}{2} \right) \sin \left( \frac{B - A}{2} \right)
\]

\[
\Rightarrow \cosh A - \cosh B = 2i \sinh \frac{A + B}{2} \sinh \frac{B - A}{2}
\]

\[
= -2 \sinh \frac{A + B}{2} \sinh \frac{B - A}{2} = 2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}
\]

\[\vdash: \text{sine} (-x) = \text{sine } x \]

7. Prove that

\[ \tanh (x + y + z) = \frac{\tanh x + \tanh y + \tanh z + \tanh x \tanh y \tanh z}{1 + \tanh x \tanh y + \tanh z \tanh x} \]

**Proof.** We know that,

\[ \tan (\alpha + \beta + \gamma) = \frac{\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma}{1 - \tan \alpha \tan \beta - \tan \beta \tan \gamma - \tan \gamma \tan \alpha} \]

Putting \( \alpha = iz; \beta = iy; \gamma = iz \), we get

\[ \tanh i(x + y + z) = \frac{\tanh (ix) + \tanh (iy) + \tanh (iz) - \tanh (ix) \tanh (iy) \tanh (iz)}{1 - \tanh (ix) \tanh (iy) - \tanh (ix) \tanh (iz) - \tanh (iy) \tanh (iz)} \]

\[ i \tanh (x + y + z) = \frac{\tanh x + i \tanh y + i \tanh z - i \tanh x \tanh y \tanh z}{1 - i \tanh x \tanh y - i \tanh z \tanh x} \]

or

\[ \tanh (x + y + z) = \frac{\tanh x + \tanh y + \tanh z + \tanh x \tanh y \tanh z}{1 + \tanh x \tanh y + \tanh z \tanh x} \]

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**ILLUSTRATIVE EXAMPLES**

**Example 1.** Separate into real and imaginary parts

(a) \( \sin (x + iy) \)

(b) \( \cos (x + iy) \)

(c) \( \tan (x + iy) \)

(d) \( \cot (x + iy) \)

(e) \( \sec (x + iy) \)

(f) \( \csc (x + iy) \)

**Sol.**

(a) \( \sin (x + iy) = \sin x \cos iy + \cos x \sin iy \)

(b) \( \cos (x + iy) = \cos x \cos iy - \sin x \sin iy \)

**Example 2.** Separate the following into real and imaginary parts:

(a) \( \sinh (x + iy) \)

(b) \( \cosh (x + iy) \)

(c) \( \tanh (x + iy) \)

(\( \vdash: \text{sech} \) \( x + iy \))

(\( \vdash: \text{coth} \) \( x + iy \))

**Sol.**

(a) \( \sinh (x + iy) = \frac{1}{i} \sin (x + iy) \)

(b) \( \cosh (x + iy) = \cos (x + iy) \)

(c) \( \tanh (x + iy) = \frac{1}{i} \tan (x + iy) \)

---

**FUNCTIONS OF A COMPLEX VARIABLE**

- (c) \( \tan (x + iy) = \frac{\sin (x + iy)}{\cos (x + iy)} \)

- (d) \( \cot (x + iy) = \frac{\cos (x + iy)}{\sin (x + iy)} \)

- (e) \( \sec (x + iy) = \frac{1}{\cos (x + iy)} \)

- (f) \( \csc (x + iy) = \frac{1}{\sin (x + iy)} \)

**Example 2.** Separate the following into real and imaginary parts:

(a) \( \sinh (x + iy) \)

(b) \( \cosh (x + iy) \)

(c) \( \tanh (x + iy) \)

(e) \( \text{sech} (x + iy) \)

(f) \( \text{coth} (x + iy) \)

**Sol.**

(a) \( \sinh (x + iy) = \frac{1}{i} \sin (x + iy) \)

(b) \( \cosh (x + iy) = \cos (x + iy) \)

(c) \( \tanh (x + iy) = \frac{1}{i} \tan (x + iy) \)

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**ILLUSTRATIVE EXAMPLES**

**Example 1.** Separate into real and imaginary parts

(a) \( \sin (x + iy) \)

(b) \( \cos (x + iy) \)

(c) \( \tan (x + iy) \)

(d) \( \cot (x + iy) \)

(e) \( \sec (x + iy) \)

(f) \( \csc (x + iy) \)

**Sol.**

(a) \( \sin (x + iy) = \sin x \cos iy + \cos x \sin iy \)

(b) \( \cos (x + iy) = \cos x \cos iy - \sin x \sin iy \)

**Example 2.** Separate the following into real and imaginary parts:

(a) \( \sinh (x + iy) \)

(b) \( \cosh (x + iy) \)

(c) \( \tanh (x + iy) \)

(e) \( \text{sech} (x + iy) \)

(f) \( \text{coth} (x + iy) \)

**Sol.**

(a) \( \sinh (x + iy) = \frac{1}{i} \sin (x + iy) \)

(b) \( \cosh (x + iy) = \cos (x + iy) \)

(c) \( \tanh (x + iy) = \frac{1}{i} \tan (x + iy) \)
FUNCTIONS OF A COMPLEX VARIABLE

\[ e^{i\theta}, e^{-i\theta} = \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} \]

By componendo and dividendo,

\[ e^{i\theta/2} - e^{-i\theta/2} = 2 \tan \frac{\theta}{2} \left( \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} \right) \]

\[ e^{i\theta/2} + e^{-i\theta/2} = 2 \left( \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} \right) \]

\[ \tanh \frac{u}{2} = \tan \frac{\theta}{2} \]

(ii)

\[ \cosh u = \frac{1 + \tan^2 \frac{\theta}{2}}{2}, \quad \sinh u = \frac{1 + \tan^2 \frac{\theta}{2}}{2} \]

[Using part (i)]

\[ \cos \theta = \sec \theta. \]

Example 4. If \( \sin (A + iB) = x + iy \), prove that

(i) \[ x^2 + y^2 = \frac{\cosh^2 B}{\sinh^2 B} \]

(ii) \[ x^2 \cos^2 A - y^2 \sin^2 A = 1. \]

Sol.

\( x + iy = \sin (A + iB) = \sin A \cos iB + \cos A \sin iB = \sin A \cosh B + i \cos A \sinh B \)

Equating real and imaginary parts on both sides,

\( x = \sin A \cosh B, y = \cos A \sinh B \)

From (i), \( \frac{x}{\cosh B} = \sin A; \quad \frac{y}{\sinh B} = \cos A \)

Squaring and adding, \( \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A = 1 \)

Also from (i), \( \frac{x}{\sin A} = \cosh B; \quad \frac{y}{\cos A} = \sinh B \)

Squaring and subtracting, \( \frac{x^2}{\cosh^2 B} - \frac{y^2}{\sinh^2 B} = \cosh^2 B - \sinh^2 B = 1 \)

or \[ x^2 \cos^2 A - y^2 \sin^2 A = 1. \]

Example 5. If \( x + iy = \cosh (u + iv) \) show that

(i) \[ \frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \]

(ii) \[ x^2 \sec^2 v - y^2 \cosec^2 v = 1. \]

Sol.

\( x + iy = \cosh (u + iv) = \cosh u \cos iv + \sinh u \sin iv \)

Equating the real and imaginary parts,

\( x = \cosh u \cos v, y = \sinh u \sin v \)

From (i), \( \frac{x}{\cosh u} = \cos v; \quad \frac{y}{\sinh u} = \sin v \)

...
Squaring and adding, \( \frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = \cos^2 v + \sin^2 v = 1 \)

From (i), \( \frac{x}{\cos v} = \cosh u; \frac{y}{\sin v} = \sinh u \)

Squaring and subtracting, \( x^2 \sec^2 v - y^2 \cosec^2 v = \cosh^2 u - \sinh^2 u = 1 \).

**Example 6.** If \( x + iy = \tan (A - iB) \), prove that

(i) \( x^2 + y^2 + 2x \cot 2A = 1 \)

(ii) \( x^2 + y^2 - 2y \coth 2B + 1 = 0 \).

**Sol.**

Changing \( i \) into \(-i\), we get \( x - iy = \tan (A + iB) \)

Now \( \tan 2A = \tan [(A + iB) + (A - iB)] \)
\[ \frac{2x}{1 - (x^2 + y^2)} \]

or \( 1 = \frac{2x}{x^2 + y^2 + 2x \cot 2A} = \frac{2x}{x^2 + y^2 + 2x \cot 2A} \)

or \(1 \cot 2A = 1 - (x^2 + y^2) = 2x \cot 2A \)

Again \( \tan (2B) = \tan [(A + iB) - (A - iB)] \)
\[ \frac{2y}{1 + (x^2 + y^2)} \]

or \( i \tanh 2B = \frac{2i}{1 + x^2 + y^2} \) or \( \coth 2B = \frac{1}{1 + x^2 + y^2} \)

Hence \( x^2 + y^2 - 2y \coth 2B + 1 = 0 \).

**Example 7.** If \( a + ib = \tanh \left( v + \frac{\pi}{4} \right) \), prove that \( a^2 + b^2 = 1 \).

**Sol.** Given \( a + ib = \tanh \left( v + \frac{\pi}{4} \right) \).

Changing \( i \) to \(-i\), we get \( a - ib = \tanh \left( v - \frac{\pi}{4} \right) \).

Multiplying (1) and (2), we have
\[ a^2 - b^2 = \tanh \left( v + \frac{\pi}{4} \right) \tanh \left( v - \frac{\pi}{4} \right) \]

\( \Rightarrow a^2 + b^2 = \frac{1}{i} \tanh \left( v + \frac{\pi}{4} \right) \tanh \left( v - \frac{\pi}{4} \right) \)

\[ = \frac{1}{i} \tanh \left( iv + \frac{\pi}{4} \right) \tanh \left( iv - \frac{\pi}{4} \right) \]

\[ = 1 \tanh \left( iv + \frac{\pi}{4} \right) \tanh \left( iv - \frac{\pi}{4} \right) \]

\[ = \frac{\sinh^2 v}{\sinh^2 v + 1} = 1 \]

**Example 8.** If \( \tan (\theta + i\phi) = \cos \alpha + i \sin \alpha \), prove that

\[ \tan \theta = \frac{\alpha}{2} \quad \text{and} \quad \phi = \frac{1}{2} \log \tan \left( \frac{\pi + \alpha}{4} \right) \] (M.D.U. May 2011)

**Sol.** \( \tan (\theta + i\phi) = \cos \alpha + i \sin \alpha \)

Changing \( i \) into \(-i\), we get

\( \tan (\theta - i\phi) = \cos \alpha - i \sin \alpha \) \( \quad \ldots (i) \)

Now \( \tan 2\theta = \tan [(\theta + i\phi) + (\theta - i\phi)] \)
\[ = \tan (\theta + i\phi) \tan (\theta - i\phi) \]
\[ = \frac{(\cos \alpha + i \sin \alpha)(\cos \alpha - i \sin \alpha)}{1 - (\cos^2 \alpha + \sin^2 \alpha)} \]
\[ = \frac{2 \cos \alpha}{1 - \cos^2 \alpha} \]
\[ = 2 \cos \alpha \]

\[ \Rightarrow 2\theta = \frac{\pi + \alpha}{2} \]

or \( \theta = \frac{\alpha}{2} \)

\[ \therefore \tan \theta = \tan \alpha \Rightarrow \theta = \pi \alpha + \alpha \]

Also \( \tan 2i\phi = \tan [(\theta + i\phi) - (\theta - i\phi)] \)
\[ = \frac{\tan (\theta + i\phi) - \tan (\theta - i\phi)}{1 + \tan (\theta + i\phi) \tan (\theta - i\phi)} \]
\[ = \frac{(\cos \alpha + i \sin \alpha) - (\cos \alpha - i \sin \alpha)}{1 + (\cos^2 \alpha + \sin^2 \alpha)} \]
\[ = 2i \sin \alpha \]

\[ \Rightarrow 2i \phi = -i \sin \alpha \quad \text{or} \quad \tan 2\phi = -\sin \alpha \]
or \[
\frac{e^{2\theta} - e^{-2\theta}}{e^{2\theta} + e^{-2\theta}} = \frac{\sin \alpha}{1} \quad \text{or} \quad \frac{e^{2\theta} + e^{-2\theta}}{e^{2\theta} - e^{-2\theta}} = \frac{1}{\sin \alpha}
\]

By componendo and dividendo,
\[
\frac{2e^{2\theta}}{2e^{2\theta}} = \frac{1 + \sin \alpha}{1 - \sin \alpha} \quad \text{or} \quad e^{2\theta} = \frac{1 + \sin \alpha}{1 - \sin \alpha}
\]

or
\[
e^{2\theta} = \frac{\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} + 2 \cos \frac{\alpha}{2} - 2 \sin \frac{\alpha}{2}}{2}
\]

or
\[
e^{2\theta} = \frac{\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}}{2} \quad \text{or} \quad e^{\theta} = \frac{\pi + \alpha}{4} \quad \text{or} \quad 2\theta = \log \tan \left( \frac{\pi + \alpha}{4} \right)
\]

Taking logarithms of both sides \( \log e^{2\theta} = \log \tan \left( \frac{\pi + \alpha}{4} \right) \) or \( 2\theta = \log \tan \left( \frac{\pi + \alpha}{4} \right) \).

\[\therefore \phi = \frac{1}{2} \log \tan \left( \frac{\pi + \alpha}{4} \right).\]

**Example 9.** Separate into real and imaginary parts \( \log \sin (x + iy) \).

**Sol.** \( \log \sin (x + iy) = \log (\sin x \cos y + i \cos x \sin y) \)
\[= \log (\sin x \cos y + i \cos x \sin y) \]
\[= \log (\alpha + i \beta), \text{ where } \alpha = \sin x \cos y, \beta = \cos x \sin y \]
\[= \frac{1}{2} \log (\alpha + \beta^2 + i \tan \frac{\beta}{\alpha}) \]
\[= \frac{1}{2} \log (\sin^2 x \cos^2 y + \cos^2 x \sin^2 y + i \tan^{-1} \frac{\cos x \sin y}{\sin x \cos y}) \]
\[= \frac{1}{2} \log \left[ \frac{1 - \cos 2x}{2} \cdot \frac{\cos 2y + 1 + \cos 2x \cdot \cos 2y - 1}{2} \right] + i \tan^{-1} (\cot x \tan y) \]
\[= \frac{1}{2} \log \left( \frac{1}{2} (2 \cos 2y - 2 \cos 2x) + i \tan^{-1} (\cot x \tan y) \right) \]
\[= \frac{1}{2} \log \left( \frac{1}{2} (2 \cos 2y - 2 \cos 2x) + i \tan^{-1} (\cot x \tan y) \right) \]

**Example 10.** If \( z = x + iy \) is a complex variable, then prove that \( \sin z \) and \( \cos z \) are not bounded.

**Sol.** We know that
\[
\sin z = \sin (x + iy) = \sin x \cos y + i \cos x \sin y
\]

\[\therefore \sin z \quad \text{and hence } \sin z \text{ is not bounded.}
\]

Similarly, \( |\cos z|^2 = \cos^2 x + \sinh^2 y \)
\[\Rightarrow |\cos z| \quad \text{and hence } \cos z \text{ is not bounded.}
\]

Hence for a complex variable \( z \), \( \sin z \) and \( \cos z \) can have any value.

**Remark.** \( |\sin z| \leq 1 \quad \text{and } |\cos z| \leq 1 \) only when \( z \) is real.

**Example 11.** Find all values of \( z \) such that
(i) \( \sinh z = 0 \)
(ii) \( \cosh z = 0 \).

**Sol.** We know that
\[
\sinh z = \sinh (x + iy) = \sinh x \cos y + i \cosh x \sin y
\]
and
\[
\cosh z = \cosh (x + iy) = \cosh x \cos y + i \sinh x \sin y
\]

\[\begin{align*}
\Rightarrow |\sinh z|^2 &= |\sinh x \cos y + i \cosh x \sin y|^2 \\
&= \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y \\
&= \sinh^2 x \cos^2 (y + \frac{\pi}{2}) + (1 + \sinh^2 x) \sin^2 y \\
&= \sinh^2 x + \sin^2 y
\end{align*}
\]

Now \( \sinh z = 0 \quad \Rightarrow \sinh z = 0 \quad \Rightarrow \sinh z = 0 \)
\[\Rightarrow \sinh^2 x + \sin^2 y = 0 \]
\[\Rightarrow \sinh x = 0 \quad \text{and} \quad \sin y = 0
\]
\[\Rightarrow \quad x = 0 \quad \text{and} \quad y = n\pi, \text{ where } n \in \mathbb{N}
\]
\[\begin{align*}
\Rightarrow \sinh z &= 0 \quad \text{only when } z \text{ is purely imaginary and } z = n\pi \text{.}
\end{align*}
\]

Similarly, \( |\cosh z|^2 = \sinh^2 x + \cos^2 y \)

Now \( \cosh z = 0 \quad \Rightarrow \cosh z = 0 \)
\[\Rightarrow \sinh^2 x + \cos^2 y = 0 \]
\[\Rightarrow \sinh x = 0 \quad \text{and} \quad \cos y = 0
\]
\[\Rightarrow \quad x = 0 \quad \text{and} \quad y = (2n + 1) \frac{\pi}{2}, \text{ where } n \in \mathbb{N}
\]
\[\begin{align*}
\Rightarrow \cosh z &= 0 \quad \text{only when } z \text{ is purely imaginary and } z = (2n + 1) \frac{\pi}{2} i.
\end{align*}
\]

**Example 12.** Find all values of \( z \) such that \( \sin z = 4 \).

**Sol.** Let \( z = x + iy \), then \( \sin z = 4 \)
\[\Rightarrow \sin (x + iy) = 4 \]
\[\Rightarrow \sin x \cos y + i \cos x \sin y = 4 \]
\[\Rightarrow \sin x \cos y = 4 \quad \text{and} \quad \cos x \sin y = 0 \]

by comparing real and imaginary parts.

From (2), we have \( \cos x = 0 \) or \( \sin y = 0 \)
\[\Rightarrow x = (2n + 1) \frac{\pi}{2}, \text{ where } n \text{ is any integer or } y = 0
\]

When \( y = 0 \), from (1), we get
\[\sin x = 4 \quad (\because \cosh 0 = 1)
\]

which is not possible since \( x \) is real.
When \( z = (2n + 1) \frac{\pi}{2} \), from (1), we get
\[
\sin \left( n \pi + \frac{\pi}{2} \right) \cosh y = 4 \quad \text{or} \quad (-1)^n \sin \frac{\pi}{2} \cosh y = 4
\]
or
\[
(-1)^n \cosh y = 4
\]
When \( n \) is odd, \( \cosh y = -4 \) which is not possible since \( \cosh y > 0 \) for every \( y \).
\[\therefore \quad n \text{ must be an even integer}.\]
From (3)
\[
cosh y = 4 \quad \text{or} \quad y = \cosh^{-1} 4
\]
Hence
\[
z = x + iy = (2n + 1) \frac{\pi}{2} + i \cosh^{-1} 4,
\]
where \( n \) is an even integer
or
\[
z = (4k + 1) \frac{\pi}{2} + i \cosh^{-1} 4,
\]
where \( k \) is any integer.

### 3.11. INVERSE HYPERBOLIC FUNCTIONS

1. Prove that \( \sinh^{-1} x = \log (x + \sqrt{x^2 + 1}) \)

**Proof.** Let \( \sinh^{-1} x = y \), then \( x = \sinh y \)
\[
\Rightarrow \quad x = \frac{e^y - e^{-y}}{2} = \frac{e^{2y} - 1}{2e^y} \Rightarrow e^{2y} - 2xe^y - 1 = 0
\]
It is a quadratic in \( e^y \)
\[\therefore e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}
\]
Rejecting the negative sign, \( e^y = x + \sqrt{x^2 + 1} \)
Taking logarithms,
\[
y = \log (x + \sqrt{x^2 + 1}) \quad \text{or} \quad \sinh^{-1} x = \log (x + \sqrt{x^2 + 1})
\]
2. Prove that \( \cosh^{-1} x = \log (x + \sqrt{x^2 - 1}) \)

**Proof.** Let \( \cosh^{-1} x = y \) then \( x = \cosh y \)
\[
x = \frac{e^y + e^{-y}}{2} = \frac{e^{2y} + 1}{2e^y} \quad \text{or} \quad e^{2y} - 2xe^y + 1 = 0
\]
It is a quadratic in \( e^y \)
\[\therefore e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}
\]
Rejecting the negative sign, \( e^y = x + \sqrt{x^2 - 1} \)
\[
y = \log (x + \sqrt{x^2 - 1}) \quad \text{or} \quad \cosh^{-1} x = \log (x + \sqrt{x^2 - 1})
\]
3. Prove that \( \tanh^{-1} x = \frac{1}{2} \log \frac{1 + x}{1 - x} \)

**Proof.** Let \( y = \tanh^{-1} x \), then \( x = \tanh y \)
\[
\text{or} \quad \frac{x}{e^y + e^{-y}} = \frac{1}{x} \quad \text{or} \quad \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^y - e^{-y}}{x}
\]

### FUNCTIONS OF A COMPLEX VARIABLE

By componendo and dividendo
\[
\frac{1 + x}{1 - x} = \frac{e^y - e^{-y}}{e^y + e^{-y}}
\]
\[\therefore \quad 2y = \log \frac{1 + x}{1 - x} \quad \text{or} \quad y = \frac{1}{2} \log \frac{1 + x}{1 - x}
\]
\[\therefore \quad \tanh^{-1} x = \frac{1}{2} \log \frac{1 + x}{1 - x}
\]

4. **Prove that:**

(i) \( \sinh^{-1} x + \sinh^{-1} y = \sinh^{-1} (x\sqrt{1 + y^2} + y\sqrt{1 + x^2}) \)

(ii) \( \cosh^{-1} x + \cosh^{-1} y = \cosh^{-1} (xy + \sqrt{x^2 - 1}(y^2 - 1)) \)

**Proof.** (i) Let \( \sinh^{-1} x = u \) and \( \sinh^{-1} y = v \)
so that
\[
x = \sinh u \quad \text{and} \quad y = \sinh v
\]
Now \( \sinh (u + v) = \sinh u \cosh v + \cosh u \sinh v \)
\[
= \sinh u \sqrt{1 + \sinh^2 v} + \sinh v \sqrt{1 + \sinh^2 u}
\]
\[
= \sqrt{1 + y^2} \quad \text{or} \quad \sinh^{-1} x + \sinh^{-1} y = \sinh^{-1} (x\sqrt{1 + y^2} + y\sqrt{1 + x^2})
\]

(ii) Let \( \cosh^{-1} x = u \) and \( \cosh^{-1} y = v \)
so that
\[
x = \cosh u \quad \text{and} \quad y = \cosh v
\]
Now, \( \cosh (u + v) = \cosh u \cosh v + \sinh u \sinh v \)
\[
= \cosh u \cosh v + \sqrt{\cosh^2 u - 1} \cdot \sqrt{\cosh^2 v - 1}
\]
\[
= xy + \sqrt{(x^2 - 1)(y^2 - 1)}
\]
\[\therefore \quad \cosh^{-1} x + \cosh^{-1} y = \cosh^{-1} (xy + \sqrt{(x^2 - 1)(y^2 - 1)})
\]

**Example 1.** Separate into real and imaginary parts

(i) \( \sin^{-1} (\cos \theta + i \sin \theta), \quad 0 < \theta < \frac{\pi}{2} \)

(ii) \( \tan^{-1} (x + iy) \).

**Sol.** (i) Let \( \sin^{-1} (\cos \theta + i \sin \theta) = x + iy \)
\[
\therefore \quad \cos \theta + i \sin \theta = \sin x \cos iy + \cos x \sin iy
\]
\[
= \sin x \cosh y + i \cos x \sinh y
\]
Equating real and imaginary parts, we have
\[ \cos \theta = \sin x \cosh y \]
\[ \sin \theta = \cos x \sinh y \]
and
\[ \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
Squaring (i) and (ii) and adding, we have
\[ 1 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \]
\[ = \sin^2 x (1 + \sinh^2 y) + (1 - \sin^2 x) \sinh^2 y = \sin^2 x + \sinh^2 y \]
or
\[ \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{From (ii),} \quad \sinh y = \sqrt{\sinh^2 y + \sin^2 x} \]
\[ \Rightarrow \cos x = \sqrt{\sin \theta} \]
\[ \therefore \text{Real part} \quad x = \cos^{-1} \left( \sqrt{\sin \theta} \right) \]
From (ii),
\[ \sinh y = \sin \theta \]
\[ \Rightarrow y = \sinh^{-1} \left( \sqrt{\sin \theta} \right) \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
\[ \therefore \sinh^2 y = 1 - \sin^2 x = \cos^2 x \]
\[ \therefore \text{Imaginary part} \quad y = \log \left( \sqrt{\sin \theta} + \sqrt{1 + \sin \theta} \right) \]
\[ \therefore \sinh^{-1} x = \log \left( x + \sqrt{x^2 + 1} \right) \]
EXERCISE 3.3

1. Prove that
   \[ (i) \cosh x + \sinh x = \cosh nx + \sinh nx; \text{ } n \text{ being a positive integer.} \]
   \[ (ii) \cosh x = \cosh 2x + \sinh 2x \]

2. If \( y = \tan x \), show that \( \sin x = \frac{1}{2} (\tan x - \cot x) \).

3. (a) If \( \tan y = \tan \alpha \tan \beta \) and \( \tan z = \cos \alpha \tan \beta \), prove that \( y = 2 \beta \text{ sec} \alpha \).
   (b) Prove that:
   \[ (i) \sin x = \sin \theta \]
   \[ (ii) \cos x = \cos \theta \]

4. If \( \tan \theta = \tan x \cos y \) and \( \tan \phi = \tan x \tan y \), prove that \( \tan \frac{\theta}{2} = \cot \frac{y}{2} \).

5. If \( \cosh (i \phi + \theta) = x + iy \), prove that
   \[ (i) x^2 \sec^2 \theta + y^2 \cosech^2 \theta = e^{2\theta} \]
   \[ (ii) x^2 \sec^2 \phi - y^2 \cosech^2 \phi = e^{2\phi} \]

6. If \( \tan (x + iy) = A + iB \), show that \( \frac{A}{B} = \frac{\sin 2x}{\sin 2y} \).

7. If \( \sin (\theta + i\phi) = \rho \cos \alpha + i \sin \alpha \), prove that
   \[ (i) \rho^2 = \frac{1}{2} (\cosh 2\phi - \cos 2\theta) \]
   \[ (ii) \tan \theta = \tan \phi \cot \theta \]

8. If \( \sin (\theta + i\phi) = \cos \alpha + i \sin \alpha \), prove that \( \cos^2 \theta = \sin \alpha \).

9. If \( \cos (\theta + i\phi) = \cos \alpha + i \sin \alpha \), prove that
   \[ (i) \sin^2 \theta = \sin \alpha \]
   \[ (ii) \cosh 2\theta + \cosh 2\phi = 2 \]

10. If \( \sin (\theta + i\phi) = \tan \alpha + i \sec \alpha \), prove that \( \cos 2\phi = 3 \).

11. If \( \sin x \cosh y = \cos \theta \) and \( \cos x \sinh y = \sin \theta \), prove that
   \[ (i) \sin^2 x = \cos^2 \theta \]
   \[ (ii) \cos^2 x = \sin^2 \theta \]

12. If \( \tan (\theta + i\phi) = \tan \alpha + i \sec \alpha \), show that \( \cos^2 \theta = \cos \alpha \).

13. If \( x = \tan x \), prove that \( \sin^2 x = \frac{\sin \theta + \cos \theta}{\sin \theta + \cos \theta} \).

14. If \( x = \tan x \), prove that \( \tan \alpha + i \sec \alpha = \tan 2x \).

15. If \( C \tan (x + iy) = A + iB \), prove that \( 2A = \frac{A}{C^2 - 1} \).

16. (a) Prove that \( \frac{1 + \cosh x + \sin x}{2} = \cosh x + \sin x \).
   (b) Prove that \( \sinh^2 x = \frac{1}{2} (\cosh x + \sinh x) \).

17. If \( \cos x = \sec \theta \), prove that
   \[ (i) \tan \frac{x}{2} = \tan \frac{\theta}{2} \]
   \[ (ii) \sin x = \sin \theta \]

18. If \( \tan \frac{x}{2} = \tan \frac{u}{2} \), prove that
   \[ (i) \cos x \cosh u = 1 \]
   \[ (ii) \tan x = \sinh u \]

19. If \( x = 2 \cos \alpha \cosh \beta \), prove that \( \sec (\alpha + i\beta) + \sec (\alpha - i\beta) = \frac{4x}{x^2 + y^2} \).

20. If \( \sin \log (A + iB) = x + iy \), show that \( \frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1 \), where \( A^2 + B^2 = e^{2\phi} \).

21. Separate into real and imaginary parts:
   \[ (i) e^{\cos (x + iy)} \]
   \[ (ii) \sin^2 (x + iy) \]
   \[ (iii) \log \cos (x + iy) \]
22. If \( \tan(x + iy) = 0 + i0 \), prove that \( \theta^2 + \phi^2 = \frac{\cosh^2 y - \cos^2 x}{\cosh^2 y - \sin^2 x} \).

23. Prove that: \( \tan \frac{u + iv}{2} = \frac{\sin u + i \sinh v}{\cos u + \cosh v} \).

24. If \( x + iy = \cos(u + iv) \), show that
   \( (i) \, (1 - x^2)^2 + y^2 = (\cosh u - \cos v)^2 \)
   \( (ii) \, x^2 + y^2 = (\cosh u + \cos v)^2 \).

25. (a) If \( \cos^{-1}(u + iv) = (x + \phi) \), prove that \( \cos^{-1} \alpha \) and \( \cosh^{-1} \beta \) are the roots of the equation
   \( x^2 - (1 + u^2 + v^2) x + u^2 = 0 \).

   (b) If \( \sin^{-1}(u + iv) = \alpha + i \beta \), prove that \( \sin^{-1} \alpha \) and \( \cosh^{-1} \beta \) are the roots of the equation
   \( x^2 - (1 + u^2 + v^2) x + u^2 = 0 \).

26. Prove that \( \tan^{-1}(e^n) = \frac{n \pi}{2} + \frac{\pi}{4} \log \frac{e^n - 1}{e^n + 1} \).

27. Find \( \tan x \) if \( 6 \sin x - \cos x = 5 \).

   [Hint. Divide both sides by \( \cosh^2 x \), square, replace \( \cosh^2 x \) by \( 1 - \tanh^2 x \) and solve for \( \tan x \).]

28. If \( \cos^{-1}(x + iy) = \alpha + i \beta \), show that
   \( (i) \, x^2 \sec^2 \alpha - y^2 \cosec^2 \beta = 1 \)
   \( (ii) \, x^2 \sec \beta + y^2 \cosech \beta = 1 \).

29. Find all the roots of the equation:
   \( (i) \, \cos x = 2 \)
   \( (ii) \, \tan x + 2 = 0 \)
   \( (iii) \, \sin x = \cosh 4 \)
   \( (iv) \, \sinh x = i \).

30. If \( \cos(u + iv) = 1 \), show that \( \sin^2 u = \sinh^2 v \).

31. If \( \cos(0 + iv) = \alpha + i \beta \), prove that
   \( (i) \, 2 \cos(0 + iv) = e^{2iv} - e^{-2iv} \)
   \( (ii) \, \cos(0 + \beta) = e^{2i\beta} \cos(0 + \beta) \).

32. Solve \( z = e^{iv} \), where \( u \) is real.

33. If \( \cosh^2(x + iy) + \sinh^2(x - iy) = \cosh^2 \alpha \), prove that \( 2(\alpha - 1)x^2 + 2(\alpha + 1)y^2 = \alpha^2 - 1 \).

Answers

21. \( (i) \, e^{x+y} \sin \, [\cos \, (\sinh x \sin y) - i \sin \, (\sinh x \sin y)] \)
   \( (ii) \, \frac{1}{2} \left[ (1 - \cos 2x \cosh 2y) + i \sin 2x \sinh 2y \right] \)
   \( (iii) \, \frac{1}{2} \left[ \cos \left( \frac{1}{2} \left( 2x + \cosh 2y \right) \right) \right] - i \tan^{-1}(\tan x \tan y) \)

27. \( \frac{4}{5} \frac{3}{5} \frac{2}{3} \frac{5}{6} \)

29. \( (i) \, 2n\pi + i \log(2 + \sqrt{3}) \)
   \( (ii) \, \log \left( \frac{n}{2} \right) + i \left( \frac{n}{2} - \frac{\pi}{2} \right) \)
   \( (iii) \, \log \left( \frac{n}{2} \right) + i \left( \frac{n}{2} - \frac{\pi}{2} \right) \)
   \( (iv) \, \log \left( \frac{n}{2} \right) + i \left( \frac{n}{2} - \frac{\pi}{2} \right) \)

32. \( z = \frac{n\pi}{2} + \frac{\pi}{4} + \frac{i}{4} \log \tan \left( \frac{\pi}{4} + \frac{\pi}{2} \right) \)

3.12. LIMIT OF A FUNCTION

Let \( f(z) \) be a single-valued function of \( z \) defined in a neighbourhood of \( z = z_0 \), then \( f(z) \) is said to have the limit \( l \) as \( z \) approaches \( z_0 \) (along any path, straight or curved) if given an arbitrary real number \( \epsilon > 0 \), however small, there exists a real number \( \delta > 0 \) such that

\[ |f(z) - l| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta \]

and we write \( \lim_{z \to z_0} f(z) = l \).

Remark 1. \( \delta \) usually depends on \( \epsilon \).

Remark 2. In real variables, \( x \to x_0 \) implies that \( x \) approaches \( x_0 \) along the number line, either from left or from right. In complex variables, \( z \to z_0 \) implies that \( z \) approaches \( z_0 \) along any path, straight or curved. The limit must be independent of the manner in which \( z \) approaches \( z_0 \) if we get two different limits as \( z \) approaches \( z_0 \) along two different paths then limit does not exist.

3.13. THEOREMS ON LIMITS

1. If \( \lim_{z \to z_0} f(z) \) exists, then it is unique.

2. If \( f(z) = u(x,y) + iv(x,y) \), where \( z = x + iy \) and \( z_0 = x_0 + iy_0 \), then \( \lim_{z \to z_0} f(z) = l = u_0 + iv_0 \)
   and only if
   \[ \lim_{x \to x_0} u(x,y) = u_0 \quad \text{and} \quad \lim_{y \to y_0} v(x,y) = v_0. \]

3. If \( \lim_{z \to z_0} f(z) = l \) and \( c \) is a constant, real or complex, then \( \lim_{z \to z_0} cf(z) = cl \).

4. If \( \lim_{z \to z_0} f(z) = l_1 \) and \( \lim_{z \to z_0} g(z) = l_2 \), then
   \( (i) \, \lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) = l_1 + l_2 \)
   \( (ii) \, \lim_{z \to z_0} [f(z) - g(z)] = \lim_{z \to z_0} f(z) - \lim_{z \to z_0} g(z) = l_1 - l_2 \)
   \( (iii) \, \lim_{z \to z_0} [fg(z)] = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z) = l_1l_2 \)
   \( (iv) \, \lim_{z \to z_0} \frac{1}{g(z)} = \frac{1}{\lim_{z \to z_0} g(z)} = \frac{1}{l_2} \quad \text{provided} \quad l_2 \neq 0 \)
   \( (v) \, \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} = \frac{l_1}{l_2} \quad \text{provided} \quad l_2 \neq 0. \)
ILLUSTRATIVE EXAMPLES

Example 1. Prove that \( \lim_{z \to 0} \frac{z}{\bar{z}} \) does not exist.

Sol. If the limit exists, then it must be independent of the manner in which \( z \) approaches 0.

Consider the path \( y \to 0 \) followed by \( x \to 0 \), we get

\[
\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{x \to 0} \frac{x - iy}{x + iy} = \lim_{x \to 0} \frac{x}{x} = 1
\]

Now consider the path \( x \to 0 \) followed by \( y \to 0 \), we get

\[
\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{y \to 0} \frac{x - iy}{x + iy} = \lim_{y \to 0} \frac{-iy}{iy} = -1
\]

As \( z \to 0 \) along two different paths, we get different limits. Hence the limit does not exist.

Example 2. Show that \( \lim_{z \to 0} \frac{\text{Re} z - \text{Im} z)^2}{|z|^2} \) does not exist.

Sol. Let \( z \to 0 \) along the path \( y = mx \), then

\[
\lim_{z \to 0} \frac{(\text{Re} z - \text{Im} z)^2}{|z|^2} = \lim_{x \to 0} \frac{(x - y)^2}{x^2 + m^2x^2} = \lim_{x \to 0} \frac{(1 - m^2)x^2}{1 + m^2}
\]

which depends on \( m \). For different values of \( m \), we have different paths and different limits. Hence the limit does not exist.

Example 3. Prove that \( \lim_{z \to -1+i} \frac{z^2 - 2z + 2}{z^2 - 2z + 2} = -\frac{1}{4} \).

Sol. \( \lim_{z \to -1+i} \frac{z^2 - 2z + 2}{z^2 - 2z + 2} = \lim_{z \to -1+i} \frac{(z - 1 + i)^2}{(z - 1 + i)^2} = \lim_{z \to -1+i} \frac{1}{(z - 1 + i)^2} = \frac{1}{(1 + 1 - 1 + 1)^2} = \frac{1}{4^2} = -\frac{1}{4} \).

FUNCTIONS OF A COMPLEX VARIABLE

EXERCISE 3.4

Show that the following limits do not exist:

1. \( \lim_{z \to 0} \frac{z}{|z|} \)
2. \( \lim_{z \to 0} \frac{z^2}{|z|^2} \)
3. \( \lim_{z \to 0} \frac{\text{Re} z}{|z|} \)
4. \( \lim_{z \to 0} \frac{z^2}{|z|^2} \)

Find the following limits:

5. \( \lim_{z \to 1} \frac{z^2 + 3z - 1}{z^2 - 2z + 4} \)
6. \( \lim_{z \to 2} \frac{z^2 + 3z - 1}{z^2 - 2z + 4} \)
7. \( \lim_{z \to 1} \frac{z^2 - 2z + 1}{z^2 - 2z + 2} \)
8. \( \lim_{z \to 1} \frac{z^2 - 2z + 1}{z^2 - 2z + 2} \)
9. \( \lim_{z \to (2z + 3i)} (z^2 - i)^2 \)
10. \( \lim_{z \to (2z + 3i)} (z^2 - i)^2 \)

Answers

5. \( -3 - 6i \)
6. \( \frac{-1 + 11i}{2} \)
7. \( 5i \)
8. \( \frac{7}{5} + \frac{4}{5}i \)
9. \( \frac{1 - i}{2} \)
10. \( -\frac{4i}{2} \)
11. \( \frac{1}{2} \)

3.14. CONTINUITY OF A FUNCTION

Let \( f(z) \) be a single-valued function of \( z \) defined in a neighbourhood of \( z = z_0 \), then \( f(z) \) is said to be continuous at \( z = z_0 \) if given an arbitrary real number \( \epsilon > 0 \), however small, there exists a real number \( \delta > 0 \) such that

\[
|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta
\]

In other words, \( f(z) \) is continuous at \( z = z_0 \) if \( \lim_{z \to z_0} f(z) = f(z_0) \).

Thus these conditions must be satisfied so that \( f(z) \) is continuous at \( z = z_0 \):
1. \( f(z) \) is defined at \( z_0 \).
2. \( f(z) \) exists at \( z = z_0 \).
3. \( \lim_{z \to z_0} f(z) = f(z_0) \).

If any of the above conditions is not satisfied then \( f(z) \) is said to be discontinuous at \( z = z_0 \).

3.15. REMOVABLE DISCONTINUITY

If \( f(z_0) \) exists and \( \lim_{z \to z_0} f(z) = l \) exists but \( f(z_0) \neq l \), then \( z = z_0 \) is called a point of removable discontinuity.
By redefining \( f(z) \) at \( z = z_0 \) such that \( f(z_0) = l \), the function can be made continuous. If \( \lim_{z \to z_0} f(z) \) does not exist, discontinuity cannot be removed. Existence of limit is the necessary condition for removable discontinuity.

### 3.16. Continuity in a Region

A function \( f(z) \) is said to be **continuous in a region** if it is continuous at all points of the region.

**Remark 1.** To examine the continuity of \( f(z) \) at \( z = x \), we put \( z = \frac{1}{w} \) and examine the continuity of \( f \left( \frac{1}{w} \right) \) at \( w = 0 \).

**Remark 2.** If \( f(z) \) and \( g(z) \) are continuous at \( z = z_0 \), then so also are the functions \( f(z) + g(z) \), \( f(z) - g(z) \), \( f(z)g(z) \) and \( \frac{f(z)}{g(z)} \), where \( g(z_0) \neq 0 \).

**Remark 3.** If \( f(z) \) is continuous in a region, then the real and imaginary parts of \( f(z) \) are also continuous in the region.

If \( f(z) = u(x, y) + i v(x, y) \), then \( f(z) \) is continuous if and only if \( u(x, y) \) and \( v(x, y) \) are separately continuous functions of \( x \) and \( y \).

### Illustrative Examples

**Example 1.** Show that the function \( f(z) \) defined by

\[
    f(z) = \begin{cases} 
        \text{Re}(z^2), & z \neq 0 \\
        \frac{1}{2}, & z = 0 
    \end{cases}
\]

is not continuous at \( z = 0 \).

**Sol.** Given \( f(0) = 0 \)

Let \( z \to 0 \) along the path \( y = mx \), then

\[
    \lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\text{Re}(z^2)}{|z|^2} = \lim_{z \to 0} \frac{z^2 - y^2}{z^2 + y^2} \]

\[
    = \lim_{z \to 0} \frac{x^2 - m^2x^2}{z^2 + m^2x^2} = \begin{cases} 
        1, & m = 0 \\
        \frac{1 - m^2}{1 + m^2}, & m \neq 0 
    \end{cases}
\]

which depends on \( m \). For different values of \( m \), we have different paths and different limits. Hence the limit does not exist and the function is not continuous at \( z = 0 \).

**Example 2.** Find the value of \( f(z) \) so that the function

\[
    f(z) = \frac{z^2 - 2z + 2}{z^2 + 2i}
\]

is continuous at \( z_0 = 1 - i \).

**Sol.**

\[
    \lim_{z \to z_0} f(z) = \lim_{z \to 1-i} \frac{z^2 - 2z + 2}{z^2 + 2i} = \lim_{z \to 1-i} \frac{(z-1)^2 + 1}{z^2 - (1-i)^2}
\]

### EXERCISE 3.5

1. Examine the continuity of the following functions:
   (i) \( f(z) = \begin{cases} 
        \frac{z^2 + 1}{z + i}, & z \neq -i \\
        0, & z = -i
    \end{cases} \) at \( z = -i \)
   (ii) \( f(z) = \begin{cases} 
        \frac{z^2 - iz^2 + z - i}{z - i}, & z \neq i \\
        0, & z = i
    \end{cases} \) at \( z = i \)
   (iii) \( f(z) = \begin{cases} 
        \frac{z^2 + 4}{z - 2i}, & z \neq 2i \\
        0, & z = 2i
    \end{cases} \) at \( z = 2i \).

2. Show that the following functions are continuous for all \( z \):
   (i) \( \sin z \)
   (ii) \( e^z \)
   (Hint: Express \( f(z) \) as \( u(x, y) + iv(x, y) \) and show that \( u(x, y) \) and \( v(x, y) \) are continuous for all real values of \( x \) and \( y \).)

3. Show that the function

\[
    f(z) = \begin{cases} 
        \frac{\text{Im}(z)}{|z|}, & z \neq 0 \\
        \frac{1}{2}, & z = 0
    \end{cases}
\]

is not continuous at \( z = 0 \).

4. Find the value of \( f(z) \) so that the function

\[
    f(z) = \frac{iz^3 - 1}{z - i}
\]

is continuous at \( z = i \).

**Answers**

1. (i) Not continuous (ii) Continuous (iii) Not continuous 4. \(-3i\)

### 3.17. Differentiability

Let \( w = f(z) \) be a single-valued function of the complex variable \( z = x + iy \). Let

\[
    w + \Delta w = f(z + \Delta z),
\]

then

\[
    \Delta w = f(z + \Delta z) - f(z)
\]

and

\[
    \frac{\Delta w}{\Delta z}
\]

is the derivative of \( f(z) \) at \( z \).
3.18. ANALYTIC FUNCTION


If a single-valued function \( f(z) \) possesses a unique derivative at every point of a region R, then \( f(z) \) is called an analytic function or a regular function or a holomorphic function of \( z \) in R.

A point where the function ceases to be analytic is called a singular point.

3.19. NECESSARY AND SUFFICIENT CONDITIONS FOR \( f(z) \) TO BE ANALYTIC

The necessary and sufficient conditions for the function \( w = f(z) = u(x, y) + iv(x, y) \) to be analytic in a region R, are:

(i) \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) and \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \) are continuous functions of \( x \) and \( y \) in the region R.

(ii) \( \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} \) and \( \frac{\partial v}{\partial y} \) are continuous functions of \( x \) and \( y \) in the region R.

The conditions in (ii) are known as Cauchy-Riemann equations or briefly CR equations.

Proof. (a) Necessary Condition. Let \( w = f(z) = u(x, y) + iv(x, y) \) be analytic in a region R, then \( \frac{dw}{dz} = f'(z) \) exists uniquely at every point of that region.

Let \( \delta x \) and \( \delta y \) be the increments in \( x \) and \( y \) respectively. Let \( \delta u \), \( \delta v \) and \( \delta x \) be the corresponding increments in \( u \), \( v \) and \( x \) respectively. Then,

\[
\lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z} = \lim_{\delta x \to 0} \left( \frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \lim_{\delta x \to 0} \left( \frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right)
\]

(b) Sufficient Condition. Let \( f(z) = u + iw \) be a single-valued function possessing partial derivatives \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \) at each point of a region R and satisfying CR equations.

We shall show that \( f(z) \) is analytic, i.e., \( f'(z) \) exists at every point of the region R.

By Taylor's theorem for functions of two variables, we have, on omitting second and higher degree terms of \( \delta x \) and \( \delta y \)

\[
\begin{align*}
f(z + \delta z) &= u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y) \\
&= [u(x, y) + \left( \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \right)] + i [v(x, y) + \left( \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y \right)] \\
&= [u(x, y) + iv(x, y)] + \left( \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \right) + \left( \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y \right)
\end{align*}
\]

or

\[
\begin{align*}
f(z + \delta x) - f(z) &= \left( \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right) \delta x + \left( \frac{\partial v}{\partial x} + i \frac{\partial v}{\partial y} \right) \delta y \\
&= \left( \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right) \delta x + \left( \frac{\partial v}{\partial x} + i \frac{\partial v}{\partial y} \right) \delta y \quad \text{[Using CR equations]}
\end{align*}
\]

Hence the necessary condition for \( f(z) \) to be analytic is that the C-R equations must be satisfied.
Functions of a Complex Variable

\[ f(z) = \frac{\partial u}{\partial x} - i \frac{\partial v}{\partial y} \]

\[ f(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y} \]

Thus \( f(z) \) exists, because \( \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} \) exist.

Hence \( f(z) \) is analytic.

Note 2. When a function \( f(z) \) is known to be analytic, it can be differentiated in the ordinary way as if \( z \) is a real variable.

Thus,

\[ f(z) = z^2 \Rightarrow f'(z) = 2z \]

\[ f(z) = \sin z \Rightarrow f'(z) = \cos z \text{ etc.} \]

3.20. Cauchy-Riemann Equations in Polar Coordinates

Let \((r, \theta)\) be the polar coordinates of the point whose cartesian coordinates are \((x, y)\), then

\[ x = r \cos \theta, \quad y = r \sin \theta, \]

\[ z = x + iy = r (\cos \theta + i \sin \theta) = re^{i\theta} \]

\[ u + iv = f(z) = f(re^{i\theta}) \]

Differentiating (1) partially w.r.t. \( r \), we have

\[ \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial \theta} = f' (re^{i\theta}). \]

Differentiating (1) partially w.r.t. \( \theta \), we have

\[ \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial r} = i f' (re^{i\theta}). \]

Or

\[ \frac{\partial u}{\partial r} = -r \frac{\partial u}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial r} \]

Equating real and imaginary parts, we get

\[ \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r} \]

Or

\[ \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = \frac{1}{r} \frac{\partial u}{\partial \theta} \]

which is the polar form of C-R equations.

3.21. Harmonic Functions

Any solution of the Laplace's equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) is called a harmonic function.

Let \( f(z) = u + iv \) be analytic in some region \( \Delta \) of the \( z \)-plane, then \( u \) and \( v \) satisfy C-R equations.
3.23. APPLICATION OF ANALYTIC FUNCTIONS TO FLOW PROBLEMS

Since the real and imaginary parts of an analytic function satisfy the Laplace's equation in two variables, these conjugate functions provide solutions to a number of field and flow problems.

For example, consider the two dimensional irrotational motion of an incompressible fluid, in planes parallel to xy-plane.

Let \( V \) be the velocity of a fluid particle, then it can be expressed as

\[
V = u_x i + u_y j
\]

Since the motion is irrotational, there exists a scalar function \( \phi(x, y) \), such that

\[
V = \nabla \phi(x, y) = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j
\]

From (1) and (2), we have

\[
u_x = \frac{\partial \phi}{\partial x} \quad \text{and} \quad \nu_y = \frac{\partial \phi}{\partial y}
\]

The scalar function \( \phi(x, y) \), which gives the velocity components, is called the velocity potential function or simply the velocity potential.

Also the fluid being incompressible, \( \text{div} \, V = 0 \)

\[
\left( \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \right) (u_x i + u_y j) = 0
\]

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0
\]

Substituting the values of \( u_x \) and \( u_y \) from (3) in (4), we get

\[
\frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial y} \right) = 0 \quad \text{or} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]

Thus the function \( \phi \) is harmonic and can be treated as real part of an analytic function

\[
u = f(z) = \phi(x, y) + i \psi(x, y)
\]

For interpretation of conjugate function \( \psi(x, y) \), the slope at any point of the curve \( \psi(x, y) = c' \) is given by

\[
\frac{\partial \psi}{\partial x} = \frac{u_y}{u_x}
\]

From (3), \( m_1 m_2 = \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} = 1 \)

This shows that the resultant velocity \( \sqrt{u_x^2 + v_y^2} \) of the fluid particle is along the tangent to the curve \( \psi(x, y) = c' \), i.e., the fluid particles move along this curve. Such curves are known as stream lines and \( \psi(x, y) \) is called the stream function. The curves represented by \( \phi(x, y) \) are called equipotential lines.

Since \( \phi(x, y) \) and \( \psi(x, y) \) are conjugate functions of analytic function \( \nu = f(z) \), the equipotential lines \( \phi(x, y) = c \) and the stream lines \( \psi(x, y) = c' \) intersect each other orthogonally.

Now,

\[
\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} (i v_x + u_y) = u_x i + \frac{\partial \psi}{\partial x} j
\]

\[
\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} (i v_x + u_y) = u_y i + \frac{\partial \psi}{\partial y} j
\]

The magnitude of resultant velocity \( \sqrt{v_x^2 + v_y^2} \),

The function \( \nu = f(z) \) which fully represents the flow pattern is called the complex potential.

In the study of electrostatics and gravitational fields, the curves \( \phi(x, y) = c \) and \( \psi(x, y) = c' \) are called equipotential lines and lines of force respectively. In heat flow problems, the curves \( \phi(x, y) = c \) and \( \psi(x, y) = c' \) are known as isothermals and heat flow lines respectively.

**ILLUSTRATIVE EXAMPLES**

**Example 1.** Find \( p \) such that the function \( f(z) \) expressed in polar coordinates as

\[
f(z) = r^2 \cos 2 \theta + ir^2 \sin 2 \theta \]

is analytic.

**Sol.** Let \( f(z) = u + iv \), then \( u = r^2 \cos 2 \theta \), \( v = r^2 \sin 2 \theta \)

\[
\frac{\partial u}{\partial r} = 2r \cos 2 \theta, \quad \frac{\partial u}{\partial \theta} = -2r^2 \sin 2 \theta
\]

\[
\frac{\partial v}{\partial r} = 2r \sin 2 \theta, \quad \frac{\partial v}{\partial \theta} = 2r^2 \cos 2 \theta
\]

For \( f(z) \) to be analytic, \( \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \) and \( \frac{\partial u}{\partial \theta} = \frac{1}{r} \frac{\partial v}{\partial r} \)

\[
2r \cos 2 \theta = pr \cos p \theta \quad \text{and} \quad 2r \sin 2 \theta = 2r \sin 2 \theta
\]

Both these equations are satisfied if \( p = 2 \).

**Note.** For a function \( f(z) \) to be analytic, the first order partial derivatives of \( u \) and \( v \) must be continuous in addition to C-R equations.
Example 2. Show that the function \( f(z) = \sqrt{|xy|} \) is not analytic at the origin, even though Cauchy-Riemann equations are satisfied there at.

U.P.T.U. 2009

Sol. Let \( f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|} \). Then \( u(x, y) = \sqrt{|xy|} \), \( v(x, y) = 0 \).

At the origin \((0, 0)\), we have
\[
\frac{du}{dx} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0
\]
\[
\frac{dv}{dy} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0
\]
\[
\frac{du}{dy} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0
\]
\[
\frac{dv}{dx} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0
\]

Clearly,
\[
\frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{du}{dy} = - \frac{dv}{dx}
\]

Hence C-R equations are satisfied at the origin.

Now \( f'(0) = \lim_{x \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|}}{x + iy} \)

If \( z \to 0 \) along the line \( y = mx \), we get
\[
f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx|^2}}{x(1 + im)} = \lim_{x \to 0} \frac{\sqrt{|m|^2}}{x + 1 + im}
\]

Now this limit is not unique since it depends on \( m \). Therefore, \( f'(0) \) does not exist.

Hence the function \( f(z) \) is not analytic at the origin.

Example 3. Prove that the function \( f(z) \) defined by
\[f(z) = \frac{x^2(1 + i) - y^2(1 - i)}{x^2 + y^2}, \quad z \neq 0 \] and \( f(0) = 0 \)

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet \( f'(0) \) does not exist.


Sol. Here
\[f(z) = \frac{(x^2 - y^2) + i(2x^2 + y^2)}{x^2 + y^2}, \quad z \neq 0 \]

Let
\[f(z) = u + iv = \frac{x^2 - y^2 + 2x^2 + y^2}{x^2 + y^2}, \quad u = \frac{x^2 - y^2}{x^2 + y^2}, \quad v = \frac{2x^2 + y^2}{x^2 + y^2}.
\]

Since \( z \neq 0 \Rightarrow x^2 + y^2 \neq 0 \)
\[u \] and \( v \) are rational functions of \( x \) and \( y \) with non-zero denominators. Thus, \( u, v \) and hence \( f(z) \) are continuous functions when \( z \neq 0 \). To test them for continuity at \( z = 0 \), on changing \( u, v \) to polar co-ordinates by putting \( x = r \cos \theta, y = r \sin \theta \), we get
\[u = r(\cos^2 \theta - \sin^2 \theta) \quad \text{and} \quad v = r(\cos^3 \theta + \sin^3 \theta)
\]

When \( z \to 0, r \to 0 \)
\[\lim_{r \to 0} u = \lim_{r \to 0} r(\cos^2 \theta - \sin^2 \theta) = 0
\]

Similarly,
\[\lim_{r \to 0} v = 0
\]
\[\therefore \lim_{z \to 0} f(z) = 0 \neq f(0)
\]
\[\Rightarrow \text{f(z) is continuous at z = 0.}
\]

Hence \( f(z) \) is continuous for all values of \( z \).

At the origin \((0, 0)\), we have
\[
\frac{du}{dx} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 1
\]
\[
\frac{dv}{dy} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = -1
\]
\[
\frac{du}{dy} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 1
\]
\[
\frac{dv}{dx} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 1
\]
\[
\therefore \frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{du}{dy} = - \frac{dv}{dx}
\]

Hence C-R equations are satisfied at the origin.

Now,
\[f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{(x^2 - y^2 + i(2x^2 + y^2)) - 0}{x + iy}
\]

Let \( z \to 0 \) along the line \( y = x \), then
\[f'(0) = \lim_{x \to 0} \frac{0 + 2ix^3}{2x(x^2 + y^2)} = \frac{i}{x} \quad \text{...(1)}
\]

Also, let \( z \to 0 \) along the \( x \)-axis (i.e., \( y = 0 \)), then
\[f'(0) = \lim_{x \to 0} \frac{x^2 + i x^3}{x^2} = 1 + i \quad \text{...(2)}
\]

Since the limits (1) and (2) are different, \( f'(0) \) does not exist.

Example 4. Prove that the function \( \sinh z \) is analytic and find its derivative.

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Sol. Here
\[f(z) = u + iv = \sinh z = \sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y
\]
\[u = \sinh x \cos y \quad \text{and} \quad v = \cosh x \sin y
\]
\[
\frac{du}{dx} = \cosh x \cos y, \quad \frac{dv}{dy} = - \sinh x \sin y
\]
\[
\frac{du}{dy} = \sinh x \sin y, \quad \frac{dv}{dx} = \cosh x \cos y
\]
\[
\therefore \frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{du}{dy} = - \frac{dv}{dx}
\]

Thus C-R equations are satisfied.
Since sinh \( x \), cosh \( x \), sin \( y \) and cos \( y \) are continuous functions, \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} \) and \( \frac{\partial v}{\partial y} \) are also continuous functions satisfying C-R equations.

Hence \( f(z) \) is analytic every where.

Now,
\[
f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \cosh x \cos y + i \sinh x \sin y = \cosh (x + iy) = \cosh z.
\]

**Example 5. Determine the analytic function whose real part is \( e^{2z} \cos 2y - y \sin 2y \).**


**Sol.** Let \( f(z) = u + iv \) be the analytic function, where \( u = e^{2z}(\cos 2y - y \sin 2y) \)

\[
\frac{\partial u}{\partial x} = 2e^{2z}(\cos 2y - y \sin 2y) + e^{2z} \cos 2y
\]

\[
= e^{2z}(2 \cos 2y - 2y \sin 2y + 2y \cos 2y)
\]

\[
\frac{\partial u}{\partial y} = e^{2z}(-2 \sin 2y - \sin 2y - 2y \cos 2y)
\]

\[
= -e^{2z}(2 \sin 2y + 2y \sin 2y + 2y \cos 2y)
\]

Since \( f(z) \) is analytic, \( u \) and \( v \) must satisfy C-R equations

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

Now

\[
\frac{\partial v}{\partial x} = e^{2z}(2 \cos 2y - 2y \sin 2y + 2y \cos 2y)
\]

Integrating w.r.t. \( y \), treating \( x \) as constant, we get

\[
u = e^{2z} \left[ 2x \sin 2y - 2y \cos 2y \right] + \frac{\sin 2y}{2} + \phi(x)
\]

where \( \phi(x) \) is an arbitrary function of \( x \).

\[
\frac{\partial u}{\partial x} = 2e^{2z}(\sin 2y + y \cos 2y) + e^{2z} (\sin 2y) + \phi'(x)
\]

But

\[
\frac{\partial u}{\partial y} = 2e^{2z}(2 \sin 2y + \sin 2y + 2y \cos 2y) + \phi'(x)
\]

\[
= 2e^{2z}(2 \sin 2y + \sin 2y + 2y \cos 2y)
\]

[From (2)]

\[
\phi'(x) = 0
\]

\[
\Rightarrow \phi(x) = c, \text{ an arbitrary constant.}
\]

\[
v = e^{2z} \left[ (x \sin 2y) + y \cos 2y \right] + c
\]

\[
f(z) = u + iv = e^{2z}(\cos 2y - y \sin 2y) + i e^{2z}(x \sin 2y + y \cos 2y) + i c
\]

\[
= e^{2z}[(x + iy) \cos 2y + i(x + iy) \sin 2y] + i c
\]

\[
= (x + iy) e^{2z} \cos 2y + i(x + iy) \sin 2y + i c
\]

\[
= z e^{2z} + i c = z e^{2z + i c} + i c
\]

**Example 6. Determine the analytic function \( u + iv \) if \( u = \log(x^2 + y^2) + x - 2y \).**

**Sol.** Here \( v = \log(x^2 + y^2) + x - 2y \)

\[
\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y
\]

\[
\frac{\partial v}{\partial x} = 2, \quad \frac{\partial v}{\partial y} = 2
\]

Since

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} - \frac{\partial u}{\partial y} \frac{\partial z}{\partial y}
\]

\[
\frac{\partial u}{\partial z} = \frac{2x}{x^2 + y^2} - \frac{2y}{x^2 + y^2} - 2
\]

[From (2)]

\[
\frac{\partial v}{\partial z} = \frac{2x}{x^2 + y^2} + i \left( \frac{2x + 1}{x^2 + y^2} \right)
\]

Replacing \( x \) by \( z \) and \( y \) by \( 0 \), we get

\[
\frac{dw}{dz} = -2 + i \left( \frac{2}{z} \right)
\]

Integrating w.r.t. \( z \), we have

\[
w = (i - 2)z + 2i \log z + c
\]
Example 8. An electrostatic fluid in the xy-plane is given by the potential function

\[ \phi = 2x^2 - y^2. \]

By C.R. equations

\[ \frac{\partial \phi}{\partial x} = 4x, \quad \frac{\partial \phi}{\partial y} = -2y. \]

So, let \( u + iv \) be a stream function.

Which expresses \( u + iv \) as an analytic function of \( z \).

Integrating \( w = e^{z} \), we have

\[ f(z) = e^z. \]

Replacing \( x \) by \( z \) and \( y \) by \( i \), we get

\[ f(z) = e^{x + iy} = e^z. \]

Since

\[ e^{z} = e^{x + iy} = e^x \cos y + ie^x \sin y, \]

which proves that \( u \) is harmonic.

Adding (1) and (2), we get

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2e^x \cos y. \]

Adding (1) and (2), we get

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2e^x \cos y. \]

\[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 2e^x \sin y. \]

Hence

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2e^x \cos y. \]

\[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 2e^x \sin y. \]

From (2) and (3), \( \delta_1 + \delta_2 = 0 \) for \( \phi = \delta_1 + \delta_2 \) and \( \delta_1 + \delta_2 = 0 \) for \( \psi = \delta_1 + \delta_2 \).

Integrating (1) w.r.t. \( x \), treating \( y \) as a constant, we get

\[ \phi = x^2 + 2xy + y^2. \]

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\[ \phi = x^2 + 2xy + y^2. \]
\[ \frac{\partial V}{\partial x} = \frac{2 \cos 2x \cosh 2y - 2 \cos 2x}{\cosh 2y - \cos 2x} = \sin 2x \]

Now, we proceed to find \( F(z) \) whose imaginary part is given.

\[ \frac{\partial V}{\partial x} = \frac{(2 \cos 2x - \cos 2x)(2 \cos 2x) - \sin 2x (2 \sin 2x)}{(\cosh 2y - \cos 2x)^2} \]
\[ = \frac{2 \cos 2x \cosh 2y - 2(\cos^3 2x + \sin^2 2x)}{(\cosh 2y - \cos 2x)^2} \]
\[ = \frac{2 \cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} \]
\[ \frac{\partial V}{\partial y} = \frac{-2 \sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} \]

Since \( f(z) \) is analytic, so is \( F(z) \).

\[ F(z) = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} + i \frac{\partial V}{\partial x} \]

By C-R equations

\[ = -2 \sin 2x \sinh 2y + i \frac{2 \cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} \]

Integrating w.r.t. \( z \), we have

\[ f(z) = \frac{2( \cos 2x - 1)}{(1 - \cos 2x)^2} \]
\[ = \frac{-2i}{1 - \cos 2x} \]
\[ = -i \cot 2x \]

Example 12. If \( f(z) \) is an analytic function with constant modulus, show that \( f(z) \) is constant.


\[ f(z) = u + iv \]

Since \( |f(z)| = constant = c \) (say), \( (c \neq 0) \)

\[ |f(z)|^2 = u^2 + v^2 = c^2 \]

Differentiating (1) partially w.r.t. \( x \) and \( y \) respectively, we have

\[ 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0 \]
\[ 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0 \]

and

\[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \]

Using Cauchy-Riemann equations

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \] and \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \)

equation (3) becomes

\[ -u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = 0 \]

Squaring and adding (2) and (4), we have

\[ (u^2 + v^2) \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] = 0 \]
or
\[ c^2 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 = 0 \]

or
\[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 = 0 \]

or
\[ |f'(z)|^2 = 0 \]

[Since \( f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \)]

\[ \therefore f'(z) = 0 \Rightarrow f(z) = \text{constant}. \]

**Example 13.** If \( f(z) \) is a regular function of \( z \), prove that

\[
\left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \left| f(z) \right|^2 = 4 \left| f'(z) \right|^2.
\]


**Sol.** Let \( f(z) = u + iv \) so that \( |f(z)| = \sqrt{u^2 + v^2} \)

or

\[ |f(z)|^2 = u^2 + v^2 \Rightarrow \phi(x, y) \text{ (say)} \]

\[ \therefore \frac{\partial \phi}{\partial x} = 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} \]

\[ \frac{\partial^2 \phi}{\partial x^2} = 2 \left[ \frac{\partial^2 u}{\partial x^2} + \left( \frac{\partial u}{\partial x} \right)^2 \right] + 2 \left[ \frac{\partial^2 v}{\partial x^2} + \left( \frac{\partial v}{\partial x} \right)^2 \right] \]

Similarly,

\[ \frac{\partial^2 \phi}{\partial y^2} = 2 \left[ \frac{\partial^2 u}{\partial y^2} + \left( \frac{\partial u}{\partial y} \right)^2 \right] + 2 \left[ \frac{\partial^2 v}{\partial y^2} + \left( \frac{\partial v}{\partial y} \right)^2 \right] \]

Adding, we get

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] + 2 \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] + 2 \left( \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \right) \]

Since \( f(z) = u + iv \) is a regular function of \( z \), \( u \) and \( v \) satisfy C-R equations and Laplace's equation.

\[ \therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]

\[ \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ \therefore \frac{\partial \phi}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \]

From (1), we get

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} \right] \]

\[ = 4 \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + \left( \frac{\partial u}{\partial x} \right)^2 \]

**Now**

\[ f(z) = u + iv \]

\[ \therefore \quad f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{and} \quad |f'(z)|^2 = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \]

From (2), we get

\[ \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \phi = 4 \left| f'(z) \right|^2 \]

or

\[ \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} \right) \phi = 4 \left| f'(z) \right|^2 \]

\[ |f(z)|^2 = 4 \left| f'(z) \right|^2. \]

**Example 14.** If \( f(z) \) is a holomorphic function of \( z \), show that

\[ \left[ \frac{\partial}{\partial x} \left| f(z) \right|^2 \right] + \left[ \frac{\partial}{\partial y} \left| f(z) \right|^2 \right] = |f'(z)|^2. \]


**Sol.** Let

\[ f(z) = u + iv \]

then

\[ |f(z)| = \sqrt{u^2 + v^2} = \phi(x, y) \text{ (say)} \]

\[ \frac{\partial \phi}{\partial x} = \frac{1}{2} \left( u + v \frac{\partial v}{\partial x} \right) \]

\[ \frac{\partial \phi}{\partial y} = \frac{1}{2} \left( u + v \frac{\partial u}{\partial y} \right) \]

Similarly,

\[ \frac{\partial \phi}{\partial x} = \frac{1}{\sqrt{u^2 + v^2}} \left( u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} \right) \]

\[ \frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{u^2 + v^2}} \left( u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} \right) \]

[Using C-R equations]

\[ \therefore \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 = \frac{1}{u^2 + v^2} \left[ \left( u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} \right)^2 + \left( -u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} \right)^2 \right] \]

\[ = \frac{1}{u^2 + v^2} \left( (u^2 + v^2) \left( \frac{\partial u}{\partial x} \right)^2 + (u^2 + v^2) \left( \frac{\partial v}{\partial x} \right)^2 \right) \]

\[ \Rightarrow \left[ \frac{\partial}{\partial x} \left| f(z) \right|^2 \right] + \left[ \frac{\partial}{\partial y} \left| f(z) \right|^2 \right] = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \]

\[ \therefore \left[ \frac{\partial}{\partial x} \left| f(z) \right|^2 \right] + \left[ \frac{\partial}{\partial y} \left| f(z) \right|^2 \right] = \left| f'(z) \right|^2. \]

Also

\[ f(z) = u + iv \]

\[ \Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \]
Example 16. If $f(z)$ is an analytic function of $z$, prove that

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log |f'(z)| = 0.
$$

Sol. We know that

$$
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \log |f'(z)| = \frac{1}{2} \left( \log |f'(z)| - \frac{1}{2} \log |f'(z)| \right)
$$

Also

$$
\log |f'(z)| = \frac{1}{2} \log |f'(z)|^2
$$

Then

$$
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \log |f'(z)| = 2 \frac{\partial}{\partial z} \left[ \log f'(z) + \log f'(\bar{z}) \right]
$$

$$
= 2 \frac{\partial}{\partial z} \left[ f'(z) \right] = 0.
$$

Note. Remember the result of Example 15.

Example 17. If $f(z)$ is an analytic function of $z$, prove that

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log |f'(z)| = 0.
$$

Sol. We know that

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log |f'(z)| = \frac{1}{2} \left( \log |f'(z)| - \frac{1}{2} \log |f'(z)| \right)
$$

Also

$$
\log |f'(z)| = \frac{1}{2} \log |f'(z)|^2
$$

Then

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log |f'(z)| = 2 \frac{\partial}{\partial z} \left[ \log f'(z) + \log f'(\bar{z}) \right]
$$

$$
= 2 \frac{\partial}{\partial z} \left[ f'(z) \right] = 0.
$$

1. (a) Determine $a$, $b$, $c$, $d$ so that the function $f(z) = (a^2 + axy + b^2y^2 + c(z^2 + d + xy^2))$ is analytic.

(b) Determine $p$ such that the function $f(z) = \frac{1}{2} \log (x^2 + y^2) + \frac{1}{2} \log (x^2 + y^2)$ is analytic.

2. (a) Show that $f(z) = ay + ib$ is everywhere continuous but not analytic.

(b) Show that $f(z) = z^2 + 2\bar{z}$ is not analytic anywhere in the complex plane. (M.D.U. Dec. 2005)

(c) if $w = \log z$, find $\frac{dw}{dz}$ and determine where $w$ is non-analytic. (J.N.T.U. 2005)

3. Show that the polar form of Cauchy-Riemann equation is

$$
\frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.
$$

4. If $f(z) = \frac{x^2 + y^2}{x^2 + y^2}$, $z \neq 0$, find $f(0) = 0$, prove that $\log \frac{1}{x} = 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

5. (a) Show that the function $f(z)$ defined by $f(z) = \frac{x^2 + y^2}{z^2 + y^2}$, $z \neq 0$, is analytic at the origin even though it satisfies Cauchy-Riemann equations at the origin.

(b) Show that $f(z) = \frac{xy}{x^2 + y^2}$, $z \neq 0, x^2 + y^2$, $z = 0$, is not analytic at $z = 0$, although C-R equations are satisfied at the origin.

(c) Examine the nature of the function

$$
f(z) = \begin{cases} 
\frac{x^2 + y^2}{z^2 + y^2}, & z \neq 0 \\
0, & z = 0 
\end{cases}
$$

in a region including the origin.

6. Determine which of the following functions are analytic:

(i) $e^z$  
(ii) $\sin z$  
(iii) $\cosh z$

(iv) $\frac{1}{z}$  
(v) $\frac{x + iy}{x^2 + y^2} + \frac{z}{z^2}$  
(vi) $\frac{1}{2} \log (x^2 + y^2) + \frac{1}{2} \log (x^2 + y^2)$

7. (a) Show that $f(z) = |z|^2$ is continuous everywhere but not analytic at any point other than the origin.

(b) Show that the function $f(z) = z$ is not analytic anywhere.

8. Show that $u + iv = \frac{z - i\alpha}{x - iy + \alpha}$, where $\alpha \neq 0$, is not an analytic function of $z = x + iy$, whereas $u - iv$ is such a function.
9. Determine the analytic function whose real part is
\[ x^2 + 3xy^2 + 3x^2 - 3y^2 + 1 \]
(M.D.U., Dec. 2008)
\[ \log(x^2 + y^2) \]
(K.U.K., Dec. 2009)
\[ e^x (x \cos y - y \sin y) \]
\[ e^x (x \sin y - y \cos y) \]
(M.D.U., Dec. 2010)
\[ (x^2 - y^2) \cos y - 2xy \sin y \]
(V.U.T.U. 2008)

10. Find the regular function whose imaginary part is
\[ \frac{x - y}{x^2 + y^2} \]
\[ e^x (x \cos y + y \sin y) \]
(U.P.T.U. 2003)
\[ e^x (x \cos y - y \sin y) \]
(M.D.U. Dec. 2006)

11. Find the real part of the regular function whose imaginary part is
\[ \frac{2 \sin x \sin y}{2x + 2y} \]
(Dombay 2006)

12. (a) Prove that \( u = x^2 - y^2 - 2xy + 2y \) is harmonic. Find the function \( v \) such that \( f(z) = u + iv \) is analytic. Also express \( f(z) \) in terms of \( z \).
(b) Show that the function \( f(x, y) = \ln(x^2 + y^2) + x - 2y \) is harmonic. Find its conjugate harmonic function \( u(x, y) \) and the corresponding analytic function \( f(z) \).
(M.D.U. May 2011)

13. An electrostatic field in the \( xy \)-plane is given by the potential function \( \phi = x^2 - y^2 \); find the stream function.

14. If \( u = f + iv \) represents the complex potential for an electric field and \( \psi = x^2 - y^2 + \frac{x}{x^2 + y^2} \) determines the function \( \phi \).

15. If the potential function is \( \log(x^2 + y^2) \), find the flux function and the complex potential function.
(M.D.U., May 2011)

16. In a two-dimensional fluid flow, the stream function is \( \psi = \tan^{-1}\left(\frac{y}{x}\right) \). Find the velocity potential \( \phi \).

17. If \( f(z) = u + iv \) is an analytic function, find \( f(z) \) if
\[ u + v = \cos(x - y) \]

18. If \( f(z) \) is an analytic function of \( z \), prove that
\[ (i) \frac{1}{(z^2 + 1)^2} \]
(M.D.U., Dec. 2008)
\[ R(f(z)) = 2 \]
\[ (i') \frac{1}{(z^2 + 1)^2} \]
(Madsen 2006)

19. Prove that \( u = \log |(x - 1)^2 + (y - 2)^2| \) is harmonic in every region which does not include the point \((1, 2)\). Find a function \( \phi \) such that \( \phi + iv \) is an analytic function of the complex variable \( z = x + iy \).
Express \( u + iv \) as a function of \( z \).

20. (a) Find the analytic function \( f(z) = u(r, \theta) + iv(r, \theta) \) such that \( u(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2 \).
(b) Find the analytic function \( f(z) = u + iv \), given \( u = |z| \cos \theta \).

21. Prove that \( u = x^2 - y^2 \) and \( v = \frac{y}{x^2 + y^2} \) are harmonic functions of \((x, y)\) but are not harmonic conjugates.

Answers
1. \( a = 2, b = -1, c = -1, d = 2 \) \( p = -1 \)
2. \( z = 0 \)
3. \( \frac{1}{z} \)
4. \( \psi = x^2 - y^2 \)
5. \( \psi = x^2 - y^2 \)
6. \( (i), (ii) \)
7. \( (i), (ii) \)
8. \( (i), (ii) \)
9. \( (i) \)
10. \( (i) \)
11. \( (i) \)
12. \( (i) \)
13. \( (i) \)
14. \( (i) \)
15. \( (i) \)
16. \( (i) \)
17. \( (i) \)
18. \( (i) \)
19. \( (i) \)
20. \( (i) \)
21. \( (i) \)

3.24. COMPLEX INTEGRATION

Let \( f(z) \) be a continuous function of the complex variable \( z = x + iy \) defined at all points of a curve C having end points A and B. Divide the curve C into \( n \) parts at the points \( A = P_0(z_0), P_1(z_1), \ldots, P_n(z_n), \ldots, P_n(z_n) = B. \)
Let $\delta z_i = z_i - z_{i-1}$ and $\xi_i$ be any point on the arc $P_{i-1}P_i$. Then the limit of the sum

$$\sum_{i=1}^{n} f(\xi_i) \delta z_i$$

as $n \to \infty$ and each $\delta z_i \to 0$, if it exists, is called the line integral of $f(z)$ along the curve $C$. It is denoted by

$$\oint_C f(z) \, dz \quad (M.D.U. \ May \ 2005)$$

In case the points $P_1$ and $P_n$ coincide so that $C$ is a closed curve, then this integral is called the contour integral and is denoted by $\oint_C f(z) \, dz$.

If $f(z) = u(x, y) + iv(x, y)$, then since $dz = dx + idy$, we have

$$\oint_C f(z) \, dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (udx - vdy) + i \int_C (vdx +udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Moreover, the value of the integral depends on the path of integration unless the integrand is analytic.

When the same path of integration is used in each integral, then

$$\oint_a f(z) \, dz = -\oint_b f(z) \, dz$$

If $c$ is a point on the arc joining $a$ and $b$, then

$$\oint_a f(z) \, dz = \oint_c f(z) \, dz + \oint_c f(z) \, dz$$

---

**ILLUSTRATIVE EXAMPLES**

**Example 1.** Evaluate $\int_0^1 (x - y + ix^2) \, dx$.

(a) along the straight line from $z = 0$ to $z = 1 + i$

(b) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + i$

(c) along the imaginary axis from $z = 0$ to $z = i$ and then along a line parallel to real axis from $z = i$ to $z = 1 + i$.

**Sol.** (a) Along the straight line $OP$ joining $O(x = 0)$ and $P(x = 1 + i)$, $y = x$, $dy = dx$ and $x$ varies from 0 to 1.

---

**Example 2.** Evaluate $\int_0^1 (x^2 - iy) \, dz$ along the paths

(a) $y = x$

(b) $y = x^2$. 

---

**Functions of a Complex Variable**

- $\int_0^1 (x - y + ix^2) \, dz = \int_0^1 (x - y + i(x^2)) \, dx + idy$

- $\int_0^1 (x^2 - x + ix^2) \, dx + idy = \int_0^1 x^2 \, dx + i \int_0^1 x^2 \, dx$

- $= (-1 + i) \left[ \frac{x^3}{3} \right]_0^1 = \left( -1 + i \right) \left( \frac{1}{3} \right) = \frac{-1}{3} + \frac{1}{3} i$

(b) Along the path $OAP$, where $A$ is $z = 1$

- $\int_0^1 (x - y + ix^2) \, dz = \int_A (x - y + ix^2) \, dz + \int_{OA} (x - y + ix^2) \, dz$

- $\int_{OA} (x - y + ix^2) \, dz = \int_0^1 (x - y + ix^2) \, dx = \left[ \frac{x^2}{2} + i \left( \frac{x^3}{3} \right) \right]_0^1 = \frac{1}{2} + \frac{1}{3} i$

- Also, along $AP$, $x = 1$, $dz = dy$ and $y$ varies from 0 to 1.

- $\int_{AP} (x - y + ix^2) \, dz = \int_0^1 (1 - y + i y) \, dy = \int_0^1 (-1 + i - 1 \frac{y^2}{2}) \, dy = -1 + i - \frac{1}{2} i$

- Hence from (1), $\int_0^1 (x - y + ix^2) \, dz = \int_0^1 (1 + i + i) \left( -1 + i - \frac{1}{2} i \right) = \frac{-1}{2} \frac{5}{6} i$

- (c) Along the path $OBP$, where $B$ is $z = i$

- $\int_0^1 (x - y + ix^2) \, dz = \int_{OB} (x - y + ix^2) \, dz + \int_{BP} (x - y + ix^2) \, dz$

- $\int_{OB} (x - y + ix^2) \, dz = \int_0^1 (x - y + ix^2) \, dx = \left[ \frac{x^3}{3} - x + i \left( \frac{x^3}{3} \right) \right]_0^1 = \frac{-1}{2} + \frac{1}{3} i$

- Also, along $AP$, $y = 1$, $dz = dx$ and $x$ varies from 0 to 1.

- $\int_{AP} (x - y + ix^2) \, dz = \int_0^1 (x - y + ix^2) \, dx = \left[ \frac{x^3}{2} + i \left( \frac{x^3}{3} \right) \right]_0^1 = \frac{1}{2} + \frac{1}{3} i$

- Hence from (2), $\int_0^1 (x - y + ix^2) \, dz = \frac{-1}{2} + \left( -\frac{1}{2} + \frac{1}{3} i \right) = \frac{-1}{2} \frac{5}{6} i$

- Note. The values of the integrals are different along the three different paths.
Sol. (a) Along the line $y = x$, 
\[ dy = dx \text{ so that } dz = dx + idx = (1 + i) dx \]
\[ \therefore \int_0^{2i} (x^2 - y^2) dz = \int_0^{2i} \left( x^2 - ix^2 \right)(1 + i) dx \]
\[ = (1 + i) \left[ \frac{x^3}{3} - \frac{x^3}{2} \right]_0^1 = (1 + i) \left( \frac{1}{3} - \frac{1}{2} \right) = \frac{1}{6} \left( 1 + i \right) \]
\[ = \frac{1}{6} + \frac{1}{6} i. \]
(b) Along the parabola $y = x^2$, $dy = 2x \, dx$ so that 
\[ dz = dx + 2ix \, dx = (1 + 2ix) \, dx \]
and $x$ varies from 0 to 1.
\[ \therefore \int_0^1 (x^2 - y^2) dz = \int_0^1 \left( x^2 - x^4 \right)(1 + 2ix) dx \]
\[ = (1 - i) \int_0^1 x^2(1 + 2ix) dx = (1 - i) \left[ \frac{x^3}{3} + i \frac{x^4}{2} \right]_0^1 \]
\[ = (1 - i) \left( \frac{1}{3} + \frac{1}{2} \right) = \frac{1}{6} + \frac{1}{6} i. \]

Example 3. Evaluate \(\int_0^{2i} (z^2) \, dz\), along 
(a) the real axis to 2 and then vertically to 2 + i. 
(b) along the line 2y = x.

Sol. 
\[ (z)^2 = (x - iy)^2 = (x^2 - y^2) - 2ixy \]
(a) Along the path OAP, where A is (2, 0) and P is (2, 1).
\[ \int_0^{2i} (z^2) \, dz \]
\[ = \int_{OA} (x^2 - y^2 - 2ixy) \, dx + \int_{AP} (x^2 - y^2 - 2ixy) \, dx \]
\[ \ldots (1) \]
Now, along OA, $y = 0$, $dz = dx$ and $x$ varies from 0 to 2
\[ \therefore \int_{OA} (x^2 - y^2 - 2ixy) \, dx = \int_0^2 x^2 \, dx = \frac{8}{3} \]
Also, along AP, $x = 2$, $dz = idy$ and $y$ varies from 0 to 1
\[ \therefore \int_{AP} (x^2 - y^2 - 2ixy) \, dy = \int_0^1 (4 - y^2 - 4iy) \, idy \]
\[ = \left[ 4iy - \frac{y^3}{3} + 2y^2 \right]_0^1 = 4i - \frac{1}{3} + 2 + \frac{11}{3} i \]
Hence from (1), we have \(\int_0^{2i} (z^2) \, dz = \frac{8}{3} + \frac{11}{3} i = \frac{14}{3} + \frac{11}{3} i.\)

Example 4. Integrate $f(z) = x^2 + iy$ from A(1, 1) to B(2, 4) along the curve $x = t, y = t^2$.
Sol. Equations of the path of integration are $x = t, \ y = t^2$
\[ \therefore \ dx = dt, \ \ dy = 2t \, dt \]
at A(1, 1), \ $t = 1$ \ and at B(2, 4), $t = 2$
\[ \therefore \int_A^B f(z) \, dz = \int_A^B (x^2 + iy)(dx + idy) = \int_A^B (t^2 + 2t^2)(dt + 2it \, dt) \]
\[ = \int_A^B (t^2 - 2it^2) dt + i \int_A^B 2t^3 dt = \left[ \frac{t^3}{3} - \frac{2t^4}{4} \right]_1^2 + i \left[ \frac{t^4}{4} \right]_1^2 \]
\[ = \frac{8}{3} + \frac{64}{5} - \frac{1}{3} - \frac{2}{5} + i (12 - \frac{3}{4}) = \frac{151}{15} + \frac{45}{4} i. \]

Example 5. Prove that 
(i) \(\oint_{C} \frac{dz}{z - a} = 2\pi i\)
(ii) \(\oint_{C} (z - a)^n \, dz = 0 \quad \text{if } n \text{ is an integer } z - 1, \text{ where } C \text{ is the circle } |z - a| = r.\)
Sol. The equation of the circle C is \(|z - a| = r \quad \text{or } z - a = re^{i\theta}\)
where $\theta$ varies from 0 to $2\pi$ as $z$ describes C once in the anti-clockwise direction.

Also \(dz = ire^{i\theta} \, d\theta\).

(i) \(\oint_{C} \frac{dz}{z - a} = i \int_0^{2\pi} \frac{ire^{i\theta}}{ire^{i\theta}} \, d\theta = i \int_0^{2\pi} d\theta = 2\pi i\)
(ii) \(\oint_{C} (z - a)^n \, dz = i \int_0^{2\pi} e^{in\theta} \cdot ire^{i\theta} \, d\theta = ire^{i\theta} \int_0^{2\pi} e^{in\theta} \, d\theta \)
\[ = ire^{i\theta} \left[ \frac{e^{in\theta}}{i(n+1)} \right]_0^{2\pi} = ir \cdot e^{i\theta} \cdot \frac{e^{i2n\pi} - 1}{n+1} \]
\[ = r \cdot e^{i\theta} \cdot \frac{e^{i2n\pi} - 1}{n+1} \]
\[ = 0. \quad \Rightarrow e^{i2n\pi} = \cos(2n\pi + 1) + i \sin(2n\pi + 1) = 1 + i(0) = 1\]
Example 6. Evaluate \[ \int_C (z^2 + 3z + 2) \, dz \] where \( C \) is the arc of the cycloid \( x = a(\theta - \cos \theta) \), 
\( y = a(1 - \cos \theta) \) between the points \((0, 0)\) and \((a\pi, 2a)\).

Sol. The function \( f(z) = z^2 + 3z + 2 \) is a polynomial and therefore analytic in \( z \)-plane. Hence the line integral of \( f(z) \) between the points \( O(0, 0) \) and \( A(a\pi, 2a) \) is independent of the path joining these points. Let us choose the path of integration as:

(i) From \((0, 0)\) to \((a\pi, 0)\) along the real axis, followed by

(ii) From \((a\pi, 0)\) to \((a\pi, 2a)\) vertically.

Thus \[ \int_C f(z) \, dz = \int_{OB} f(z) \, dz + \int_{BA} f(z) \, dz \]

Now along \( OB, y = 0 \):

\[ z = x + iy = x, \quad dz = dx \]

and \( x \) varies from 0 to \( a\pi \).

\[ f(z) = x^2 + 3x + 2 = x^2 + 3x + 2 \]

\[ \int_{OB} f(z) \, dz = \int_0^{a\pi} (x^2 + 3x + 2) \, dx = \left[ \frac{x^3}{3} + \frac{3x^2}{2} + 2x \right]_0^{a\pi} \]

\[ = \frac{\pi^3 a^3}{3} + \frac{3\pi^2 a^2}{2} + 2a\pi = \frac{a\pi}{6} (2\pi^2 a^2 + 9\pi a + 12) \]

Also, along \( BA, x = a\pi \):

\[ z = x + iy = a\pi + iy, \]

\[ dz = idy \text{ and } y \text{ varies from } 0 \text{ to } 2a. \]

\[ f(z) = (a\pi + iy)^2 + 3(a\pi + iy) + 2 \]

\[ \int_{BA} f(z) \, dz = \int_0^{2a} \left[ (a\pi + iy)^2 + 3(a\pi + iy) + 2 \right] \, idy \]

\[ = \left[ \left( \frac{(a\pi + iy)^2}{2i} + \frac{3(a\pi + iy)}{2} + 2i \right) \right]_0^{2a} \]

\[ = \frac{1}{3} \left( \frac{(a\pi + i2a)^2}{2} \right) + \frac{3}{2} \left( a\pi + i2a \right) + 2i \cdot 2a - \frac{1}{3} \pi^3 a^3 - \frac{3}{2} \pi^2 a^2 \]

\[ = \frac{a^3}{3} (\pi + 2i)^3 + \frac{3a^2}{2} (\pi + 2i)^2 + 4ia \cdot \frac{1}{3} \pi^3 a^3 - \frac{3}{2} \pi^2 a^2 \]

From (1), we have

\[ \int_C (z^2 + 3z + 2) \, dz = \frac{a\pi}{6} (2\pi^2 a^2 + 9\pi a + 12) + \frac{a^3}{3} \pi^3 a^3 + \frac{3a^2}{2} (\pi + 2i)^2 + 4ia \cdot \frac{1}{3} \pi^3 a^3 - \frac{3}{2} \pi^2 a^2 \]

\[ = \frac{a^3}{3} (\pi + 2i)^3 + \frac{3a^2}{2} (\pi + 2i)^2 + 2a(\pi + 2i) \]

\[ = \frac{a^3}{3} (\pi + 2i)^3 + \frac{3a^2}{2} (\pi + 2i)^2 + 2a(\pi + 2i) \]

**EXERCISE 3.7**

1. Evaluate \[ \int_0^{2a} z^2 \, dz \text{ along } \]

   (a) the line \( y = \frac{z}{3} \)

   (c) the parabola \( x = 3y^2 \).

2. Evaluate \[ \int_{-1+i}^{2+i} (2x + iy + 1) \, dz \text{ along } \]

   (a) the straight line joining \((1 - i)\) to \((2 + i)\)

   (b) the curve \( x = t^2, y = 2t^2 - 1 \).

3. Evaluate \[ \int_{-1+i}^{2+i} z \, dz \text{ along the curve given by } z = t^2 + it. \]  

   (M.D.U. Dec. 2006)

4. Evaluate \[ \int_C |z|^2 \, dz \text{ around the square with vertices at } (0, 0), (1, 0), (1, 1) \text{ and } (0, 1). \]

5. Show that \[ \int_C (z + 1) \, dz = 0 \text{, where } C \text{ is the boundary of the square whose vertices are at the points } z = 0, z = 1, z = 1 + i \text{ and } z = i. \]  


6. Evaluate \[ \int_C (y - x - 3x^2) \, dz \text{ where } C \text{ is the straight line from } z = 0 \text{ to } z = 1 + i. \]

7. (a) Evaluate \[ \int_C (z - z^2) \, dz \text{, where } C \text{ is the upper half of the circle } |z| = 1. \text{ What is the value of this integral if } C \text{ is the lower half of the circle?} \]

   (b) Evaluate \[ \int_C (z - z^2) \, dz \text{, where } C \text{ is the upper half of the circle } |z| = 1. \]  

   (M.D.U. Dec. 2013)

   (c) Evaluate \[ \int_C (z - z^2) \, dz \text{ where } C \text{ is the upper half of the circle } |z - 2| = 3. \text{ What is the value of the integral if } C \text{ is the lower half of the above given circle?} \]  

   (M.D.U. Dec. 2009)

8. Prove that \[ \int_C z \, dz = -\pi i \text{ or } \pi i \text{ according as } C \text{ is the semi-circular arc } |z| = 1 \text{ from } -1 \text{ to } 1 \text{ above} \]

   or below the real axis.  

   (M.D.U. May 2005)

9. Show that for every path between the limits, \[ \int_{-1}^{2+i} (2z + z^2) \, dz = \frac{i}{3} \]

10. Evaluate \[ \int_{-1}^{2+i} z^3 \, dz \text{ along the line joining the points } (1, -1) \text{ and } (2, 3). \]
11. Evaluate $\int_C dz$, where $C$ is the contour
(a) straight line from $z = i$ to $z = 0$,
(b) left half of the unit circle $|z| = 1$ from $z = i$ to $z = 0$,
(c) circle given by $|z + 1| = 1$ described in the clockwise sense.

12. Evaluate $\int_C \log |z| dz$, where $C$ is the unit circle $|z| = 1$.

Answers
1. (a) $\frac{25}{3}$
2. (a) $4 + 8i$
3. (a) $10 - 8i$
4. (a) $-1 + i$
5. (a) $\frac{2}{3}$
6. (a) $0$
7. (a) $-\frac{2}{3}$
8. (a) $\frac{8}{3}i$
9. (a) $\frac{1}{6}$
10. (a) $i$
11. (a) $2i$
12. $2\pi i$

3.25. SIMPLY AND MULTIPLY CONNECTED REGIONS

A curve is called simple closed curve if it does not cross itself (Fig. 1). A curve which crosses itself is called a multiple curve (Fig. 2).

![Fig 1](image1)

![Fig 2](image2)

A region is called simply connected if every closed curve in the region encloses points of the region only, i.e., every closed curve lying in it can be contracted indefinitely without passing out of it. A region which is not simply connected is called a multiply connected region. In plain terms, a simply connected region is one which has no holes. Fig. 3 shows a multiply connected region $R$ enclosed between two separate curves $C_1$ and $C_2$. (There can be more than two separate curves. We can convert a multiply connected region into a simply connected one, by giving it one or more cuts (e.g., along the dotted line $AB$).

![Fig 3](image3)

3.26. CAUCHY'S INTEGRAL THEOREM

Statement. If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve $C$, then
$$\int_C f(z)dz = 0$$

Proof.

Let $R$ be the region bounded by the curve $C$.

Let $f(z) = u(x, y) + iv(x, y)$, then
$$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C (u + iv)dx + \int_C (u + iv)idy$$

Since $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are also continuous in $R$. Hence by Green's Theorem, we have
$$\int_C f(z)dz = \int_R \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \int_R \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

Now $f(z)$ being analytic at each point of the region $R$, by Cauchy-Riemann equations, we have
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, the two double integrals in (2) vanish.

Hence
$$\int_C f(z)dz = 0$$

Cor. 1. If $f(z)$ is analytic in a region $R$ and $P$ and $Q$ are two points in $R$, then $\int_P^Q f(z)dz$ is independent of the path joining $P$ and $Q$ and lying entirely in $R$.

Let $PAB$ and $PBC$ be any two paths joining $P$ and $Q$.

By Cauchy's theorem,$$\int_{PAB} f(z)dz = 0$$
$$\int_{PBC} f(z)dz = 0$$

Thus
$$\int_{PAB} f(z)dz - \int_{PBC} f(z)dz = 0$$

Hence
$$\int_{PAC} f(z)dz = \int_{PBC} f(z)dz$$

Cor. 2. If $f(z)$ is analytic in the region bounded by two simple closed curves $C$ and $C_1$, then
$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

Let $AB$ be a cross-cut joining the curves $C$ and $C_1$, then the doubly connected region becomes simply connected.

By Cauchy's theorem,$$\int_{AWCA} f(z)dz = 0$$
$$\int_{BC} f(z)dz = 0$$

(integrals around a closed curve are taken positive when the curve is traversed in counter-clockwise direction).
Hence \[ \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz \]

The theorem can be extended.

If a closed curve \( C \) contains non-intersecting closed curves \( C_1, C_2, \ldots, C_n \), then by introducing cross-cuts, it can be shown that

\[ \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz. \]  

(M.D.U. May 2006)

3.27. CAUCHY'S INTEGRAL FORMULA


**Statement.** If \( f(z) \) is analytic within and on a closed curve \( C \) and \( a \) is any point within \( C \), then

\[ f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz. \]

**Proof.** Consider the function \( \frac{f(z)}{z-a} \), which is analytic at every point within \( C \) except at \( z = a \).

Draw a circle \( C_1 \) with \( a \) as centre and radius \( \rho \) such that \( C_1 \) lies entirely inside \( C \).

Thus \( \frac{f(z)}{z-a} \) is analytic in the region between \( C \) and \( C_1 \).

\( \therefore \) By Cauchy's theorem, we have

\[ \oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz \]

Now, the equation of circle \( C_1 \) is

\[ |z-a| = \rho \text{ or } z-a = \rho e^{i\theta} \]

so that

\[ dz = i\rho e^{i\theta} d\theta \]

Hence by (1), we have

\[ \oint_{C_1} \frac{f(z)}{z-a}\,dz = i \oint_0^{2\pi} f(a+\rho e^{i\theta})\,d\theta \]

Hence by (1), we have

\[ \oint_C \frac{f(z)}{z-a}\,dz = i \oint_0^{2\pi} f(a+\rho e^{i\theta})\,d\theta = i \oint_0^{2\pi} f(a)\,d\theta = 2\pi if(a) \]

Hence

\[ f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz \]

which is the required Cauchy's integral formula.

**Cor.** By Cauchy's integral formula, we have

\[ f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz \]

where \( a \) is any point within \( C \) and may be treated as a parameter.

**ILLUSTRATIVE EXAMPLES**

**Example 1.** Evaluate \( \oint_C (z^2 - y^2 + 2ixy)\,dz \), where \( C \) is the contour \( |z| = 1 \).

**Sol.** \( f(z) = z^2 - y^2 + 2ixy = z^2 \) is analytic everywhere within and on \( |z| = 1 \).

\( \therefore \) By Cauchy's integral theorem, \( \oint_C f(z)\,dz = 0 \).

**Example 2.** Evaluate \( \oint_C (3z^2 + 4z + 1)\,dz \), where \( C \)

is the arc of the cycloid \( x = a(\theta - \sin \theta), y = a(1 - \cos \theta) \)

between \((0, 0)\) and \((2\pi, 0)\).

**Sol.** Here, \( f(z) = 3z^2 + 4z + 1 \) is analytic everywhere so that the integral is independent of the path of integration and depends only on the end points \( z_1 = 0 + i0 \) and \( z_2 = 2\pi + i0 \).

\[ \oint_C (3z^2 + 4z + 1)\,dz = \oint_0^{2\pi} (3z^2 + 4z + 1)\,dz \]

\[ = [z^3 + 2z^2 + z]_0^{2\pi} = 8\pi^3 a^3 + 8\pi^2 a^2 + 2\pi a (4\pi^2 a^2 + 4\pi a + 1) \]

**Example 3.** Evaluate \( \oint_C \frac{z^2}{z+1}\,dz \), where \( C \) is the circle

\( (a) \ |z| = 2 \) \quad \( (b) \ |z| = \frac{1}{2} \).

**Sol.** \( f(z) = z^2 \) is an analytic function.

\( (a) \) The point \( a = -1 \) lies inside the circle \( |z| = 2 \).

\( \therefore \) By Cauchy's integral formula,

\[ \oint_C \frac{f(z)}{z+1}\,dz = f(-1) = \oint_C \frac{z^2}{z+1}\,dz = 2\pi i \]
(b) The point \( a = -1 \) lies outside the circle \( |z| = \frac{1}{2} \).

The function \( \frac{e^{iz}}{z+1} \) is analytic within and on \( C \).

By Cauchy's integral theorem, we have

\[ \oint_C \frac{e^{iz}}{z+1} \, dz = 0. \]

**Example 4. Evaluate** \[ \oint_C \frac{\cos \pi z}{z^2 - 1} \, dz, \text{ where } C \text{ is the circle } |z| = 3. \]

**Sol.** The integrand has singularities where \( z^2 - 1 = 0 \) i.e., \( z = 1 \) and \( z = -1 \).

Both these points lie within the circle \( |z| = 3 \).

\( f(z) = \cos \pi z \) is an analytic function.

\[ \frac{1}{2} \int_C \cos \pi z \, dz - \frac{1}{2} \int_C \cos \pi z \, dz = -2\pi i \left( \frac{1}{2} \int_C \cos \pi z \, dz + \frac{1}{2} \int_C \cos \pi z \, dz \right) = -2\pi i \cos \pi + 2\pi i \cos \pi = -2\pi i (\cos \pi + \cos \pi) = -4\pi i. \]

\[ \int_C \frac{1}{z+1} \, dz = \begin{cases} \pi i & \text{if } C \text{ encloses } z = 1, \\ -\pi i & \text{if } C \text{ encloses } z = -1. \end{cases} \]

**Example 5. Evaluate** \[ \oint_C \frac{z^2 + z}{z^2 - 1} \, dz, \text{ where } C \text{ is the circle } |z-1| = 1. \]

**Sol.** The integrand has singularities where \( z^2 - 1 = 0 \) i.e., \( z = 1 \) and \( z = -1 \).

The circle \( |z-1| = 1 \) has centre at \( z = 1 \) and radius 1 and includes the point \( z = 1 \).

\( f(z) = z^2 + z \) is analytic within and on \( C \).

\[ \frac{1}{2} \int_C \frac{z^2 + z}{z^2 - 1} \, dz = \frac{1}{2} \int_C \frac{z^2 + z}{z^2 - 1} \, dz = \frac{1}{2} \int_C \frac{z^2 + z}{z^2 - 1} \, dz = 2\pi i \]

By Cauchy's integral formula,

\[ \oint_C \frac{z^2 + z}{z^2 - 1} \, dz = 2\pi i \left( \frac{1}{2} \int_C \frac{z^2 + z}{z^2 - 1} \, dz \right). \]

**Exercise 3.8**

1. Verify Cauchy's theorem for the integral of \( z^2 \) taken over the boundary of the
   (i) rectangle with vertices \(-1, 1 + i, -1 + i, 1 + i, -1 + i, 1, 1 + i, -1, -1 + i\).
   (ii) triangle with vertices \((1, 2), (1, 0), (3, 2)\).
2. (a) Evaluate \( \int_C \frac{dz}{(z-a)^n} \), \( n = 2, 3, 4 \), where \( C \) is a closed curve containing the point \( z = a \).

(b) Evaluate \( \int_C \frac{z^3 - z + 1}{z - 1} \, dz \), where \( C \) is the circle \( |z| = \frac{1}{2} \).

(M.D.U. Dec. 2011)

3. Evaluate \( \int_C \frac{z^3 + 5}{z^2 - 3} \, dz \), where \( C \) is the circle.

(a) \( |z| = 4 \)

(b) \( |z| = 1 \).

4. Evaluate \( \int_C \frac{e^{z+2}}{z-2} \, dz \), where \( C \) is the circle.

(a) \( |z| = 3 \)

(b) \( |z| = 1 \).

5. Evaluate \( \int_C \frac{3z^2 + 7z + 1}{z + 1} \, dz \), where \( C \) is the circle.

(a) \( |z| = 1 \)

(b) \( |z+i| = 1 \)

(c) \( |z| = \frac{1}{2} \).

6. Evaluate \( \int_C \frac{3z^2}{z^2 + 5} \, dz \), where \( C \) is the circle \( |z| = 5 \).

7. Evaluate \( \int_C \frac{z^2 + z + 1}{z^2 - 3z + 2} \, dz \), where \( C \) is the ellipse \( 4x^2 + 9y^2 = 1 \).


8. Evaluate \( \int_C \frac{e^{z+2}}{z^2 - 1} \, dz \) around a rectangle with vertices

(a) \( 2+i, 2-i, -2+i, -2-i \).

(M.D.U. May 2006)

(i) Evaluate \( \int_C \frac{e^z}{z-3} \, dz \), where \( C \) is the circle \( |z| = 1 \).

(M.D.U. May 2006)

(ii) Evaluate \( \int_C \frac{e^{z^2}}{z^2 + 2} \, dz \), where \( C \) is the circle \( |z| = 4 \).

(M.D.U. Dec. 2010)

(iii) Evaluate \( \int_C \frac{e^z}{z} \, dz \), where \( C \) is the circle \( |z| = 1 \).

(M.D.U. Dec. 2009)

(iv) Evaluate \( \int_C \frac{e^z}{z - 2i} \, dz \), where \( C \) is the square with vertices at \( 1, 1+i, -1+i, -1, -1, -1+i \).

(M.D.U. Dec. 2009)

Answers

2. (a) 0

(b) 0

3. (a) 2\( \pi \)

(b) 0

4. (a) 2\( \pi \) \( e^z \)

(b) 0

5. (a) -6\( \pi \)

(b) 0

(c) 0

6. 2\( \pi \)

7. 0

8. (a) 0

(b) \( -\pi i \)

9. (i) \( \pi i \)

(ii) 16 \( \pi i \)

(iii) 12 \( \pi i \)

10. (a) 2\( \pi \) \( i \)

(b) -\( \pi i \) 

11. 12 \( \pi i \)

12. \( -\frac{\pi i}{2} \)

13. 2\( \pi i \)

14. 2\( \pi i \)

15. 18\( \pi i \) \( -2 \)

16. \( -2 \) \( \pi i \)

17. 2\( \pi i \)

18. (a) 4\( \pi i \)

(b) 2\( \pi i \)

19. 20\( \pi i \) \( -1 \) \( -14 \) \( \pi i \) \( 10 \) \( \pi i \)

20. (i) \( \frac{\pi i}{e} \)

(ii) \( 4 \pi i \)

(iii) \( 2 \pi i / e \)

(i) \( \frac{i}{4} \)

(ii) \( -\pi i \)

(iii) 0

(iv) 72\( \pi i \) \( \ln 2 = 0.6931 \)
3.28. TRANSFORMATION OR MAPPING

We know that the real function \( y = f(x) \) can be represented graphically by a curve in the \( xy \)-plane. Also, the real function \( z = f(x, y) \) can be represented by a surface in three-dimensional space. However, this method or graphical representation fails in the case of complex functions because a complex function \( w = f(z) \), i.e., \( u + iv = f(x + iy) \), involves four real variables, two independent variables \( x, y \) and two dependent variables \( u, v \). Thus a four-dimensional region is required to represent it graphically in the Cartesian fashion. As it is not possible, we choose two complex planes and call them \( z \)-plane and \( w \)-plane. In the \( z \)-plane, we plot the point \( z = x + iy \) and in the \( w \)-plane, we plot the corresponding point \( w = u + iv \). Thus the function \( w = f(z) \) defines a correspondence between points of these two planes. If the point \( z \) describes some curve \( C \) in the \( z \)-plane, the point \( w \) will move along a corresponding curve \( C' \) in the \( w \)-plane, since to each \( (x, y) \) there corresponds a point \( (u, v) \). The function \( w = f(z) \) thus defines a mapping or transformation of the \( z \)-plane into the \( w \)-plane.

For example, consider the transformation \( w = z + (1 - i) \). Let us determine the region \( D' \) of the \( w \)-plane corresponding to the rectangular region \( D \) in the \( z \)-plane bounded by \( x = 0, y = 0, x = 1 \) and \( y = 2 \).

Since \( w = z + (1 - i) \), we have

\[
\begin{align*}
u + iv &= (x + iy) + (1 - i) = (x + 1) + i(y - 1) \\u &= x + 1 \quad \text{and} \quad v &= y - 1
\end{align*}
\]

Hence the lines \( x = 0, y = 0, x = 1 \) and \( y = 2 \) in the \( z \)-plane are mapped onto the lines \( u = 1, v = 1, u = 2 \) and \( v = 1 \) in the \( w \)-plane. The regions \( D \) and \( D' \) are shown shaded in the figure.

"This part is not included in the syllabus of M.D.U., Rohatk."
Similarly, \( \delta w = r'e^{i\theta} \), where \( r' \) is the modulus and \( \theta' \) is the amplitude of \( \delta w \).

Let the tangent to \( C \) at \( P \) make an angle \( \alpha \) with the \( x \)-axis and the tangent to \( C' \) at \( P' \) make an angle \( \alpha' \) with the \( u \)-axis, then as \( \delta x \to 0, \theta \to \alpha \) and \( \theta' \to \alpha' \).

\[
\frac{\delta w}{\delta x} = \frac{r'}{r} e^{i \theta'}
\]

\( f'(z) = \frac{dw}{dz} = \lim_{\delta x \to 0} \frac{\delta w}{\delta x} = \lim_{\delta x \to 0} \frac{r'}{r} e^{i \theta'} \) \hspace{1cm} (1)

Since \( f'(z) \neq 0 \), let \( f'(z) = re^{i\phi} \), then \( \rho = \left| f'(z) \right| \) and \( \phi \) = amplitude of \( f'(z) \).

From (1), \( \rho = \lim_{\delta x \to 0} \frac{r'}{r} e^{i \theta'} \)

Thus
\[
\rho = \lim_{\delta x \to 0} \frac{r'}{r} e^{i \theta'}
\]

and
\[
\phi = \lim_{\delta x \to 0} \theta' - \alpha = \phi' - \alpha \] \hspace{1cm} (2)

Now let \( C_1 \) be another curve through \( P \) in the \( z \)-plane and \( C'_1 \) the corresponding curve through \( P' \) in the \( u \)-plane. If the tangent to \( C_1 \) at \( P \) makes an angle \( \beta \) with the \( x \)-axis and the tangent to \( C'_1 \) at \( P' \) makes an angle \( \beta' \) with the \( u \)-axis, then as in (3), \( \phi = \beta' - \beta \)

\[
(3) \hspace{1cm} \alpha' - \alpha = \beta' - \beta \ or \ \beta - \alpha = \beta' - \alpha = \gamma
\]

From (3) and (4), \( \alpha' - \alpha = \beta' - \beta \ or \ \beta - \alpha = \beta' - \alpha = \gamma \) \hspace{1cm} (4)

Thus angle between the curves before and after the mapping is preserved in magnitude and sense. Hence the mapping by the analytic function \( w = f(z) \) is conformal at each point where \( f'(z) \neq 0 \).

Note 1. A point at which \( f'(z) = 0 \) is called a critical point of the transformation.

Note 2. From (2) \( \rho = \lim_{\delta x \to 0} \frac{r'}{r} e^{i \theta'} \)

It follows that under the conformal transformation \( w = f(z) \), the lengths of arcs through \( P \) are magnified in the ratio \( \rho \); where \( \rho = \left| f'(z) \right| \). Thus an infinitesimal length in the \( z \)-plane is magnified by the factor \( \left| f'(z) \right| \) in the \( u \)-plane and consequently infinitesimal area in the \( z \)-plane are magnified by the factor \( \left| f'(z) \right| \) in the \( u \)-plane.

Note 3. From (3), \( \alpha' = \alpha + \phi \) shows that the tangent to the curve \( C \) at \( P \) is rotated through an angle \( \phi \) under the given transformation.

Note 4. A harmonic function remains harmonic under a conformal transformation.

### Functions of a Complex Variable

#### 3.31. COEFFICIENT OF MAGNIFICATION

Coefficient of magnification for the conformal transformation \( w = f(z) \) at \( z = \alpha + i \beta \) is given by \( \left| f'(\alpha + i \beta) \right| \) where \( \left| \cdot \right| \) denotes absolute value.

#### 3.32. ANGLE OF ROTATION

Angle of rotation for the conformal transformation \( w = f(z) \) at \( z = \alpha + i \beta \) is given by \( \arg f'(\alpha + i \beta) \).

**Example 1.** For the conformal transformation \( w = z^2 \), show that

(i) the coefficient of magnification at \( z = 1 + i \) is \( 2 \sqrt{2} \).

(ii) The angle of rotation at \( z = 1 + i \) is \( \pi/4 \).

**Solution.**

\( f(z) = z^2 \) \hspace{1cm} \( f'(z) = 2z \) and \( f'(1 + i) = 2 + 2i \).

The coefficient of magnification at \( z = 1 + i \) is \( |f'(1 + i)| = \sqrt{4 + 4} = 2 \sqrt{2} \).

The angle of rotation at \( z = 1 + i \) is \( \arg f'(1 + i) = -\frac{\pi}{4} \).

#### 3.33. SOME STANDARD TRANSFORMATIONS

1. Translation: \( w = z + c \), where \( c \) is a complex constant

Let \( z = x + iy, \ c = a + ib \) and \( w = u + iv \)

then the transformation becomes \( u + iv = (x + iy) + (a + ib) = (x + a) + iy(b) \).

so that \( u = x + a \) and \( v = y + b \).

Thus the transformation is a mere translation of the axes and preserves the shape and size.

For example, the rectangle OMPN in \( z \)-plane is transformed to rectangle O'M'P'N' in the \( w \)-plane under the transformation \( w = z + (1 + 2i) \).

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
2. Rotation and Magnification: \( w = cx \), where \( c \) is complex constant.

Let \( c = re^{i\theta} \) and \( w = Re^{i\phi} \), then the transformation becomes \( Re^{i\phi} = \rho r e^{i(\theta + \alpha)} \).

\[ R = \rho r \quad \text{and} \quad \phi = \theta + \alpha \]

Thus the transformation maps a point \((r, \theta)\) in the \( z \)-plane into a point \((\rho r, \theta + \alpha)\) in the \( w \)-plane. Hence the transformation consists of magnification of the radius vector of \( P \) by \( \rho = |c| \) and its rotation through an angle \( \alpha = \text{amp}(c) \).

Thus under this transformation figure in \( w \)-plane is similar to the figure in \( z \)-plane (magnified by \( |c| \)) but rotated through an angle \( \alpha \).

**Note 1.** If \( \alpha > 0 \) then rotation is anticlockwise and if \( \alpha < 0 \) then rotation is clockwise.

**Note 2.** If \( w = cz \), if \( \alpha \) is real then \( \alpha = 0 \), then this transformation is only that of magnification (no rotation).

In this case the two figures in \( z \)-plane and \( w \)-plane are similarly situated about their respective origins but figure in \( w \)-plane is \( |c| \) times figure in \( z \)-plane. Such mapping is called Magnification.

For example the transformation \( w = (1 + i)z \) maps the square OMPN bounded by \( x = 0, y = 0, y = 1, x = 1 \) in \( z \)-plane to the square OMP'N' in \( w \)-plane.

![Diagram](image)

**Verification:** Here \( i = 1 + i \), \( |c| = \sqrt{2} \).

Each side of the square in \( w \)-plane is \( |c| \) times \( (i.e., \sqrt{2}) \) the side of the square in \( z \)-plane also amp of \( c = \text{tan}^{-1} \frac{1}{1} = \frac{\pi}{4} \), and the sides of the square are rotated through an angle \( \frac{\pi}{4} \).

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**FUNCTIONS OF A COMPLEX VARIABLE**

This transformation is rotation as well as magnification.

Consider another example in which \( c \) is real:

**Example:** The transformation \( w = 2z \) maps the triangular region OAB bounded by the lines \( x = 0, y = 0, x + y = 1 \) into a similar triangle OA'B' in \( w \)-plane.

![Diagram](image)

Here \( w = 2z \)

\( u = 2x \)
\( v = 2y \)
\( x = 0 \) maps into \( u = 0 \)
\( y = 0 \) maps into \( v = 0 \)

\( x + y = 1 \) maps into \( u + v = 1 \) i.e., \( u + v = 2 \)

**Verification:** The two figures are similar but figure in \( w \)-plane is \( |c| \) times figure in \( z \)-plane.

This transformation is only Magnification.

**3. Inversion:** \( w = \frac{1}{z} \)

Let \( z = re^{i\theta} \) and \( w = Re^{i\phi} \), then the transformation becomes \( Re^{i\phi} = \frac{1}{r} e^{-i\theta} \), so that \( R = \frac{1}{r} \) and \( \phi = -\theta \).

Thus under the transformation \( w = \frac{1}{z} \), a point \( P(r, \theta) \) in \( z \)-plane is mapped into the point \( P' \left( \frac{1}{r}, -\theta \right) \).

Consider the \( w \)-plane superimposed on the \( z \)-plane. If \( P \) is \((r, \theta)\) and \( P_1 \) is \( \left( \frac{1}{r}, -\theta \right) \), then

\[ OP_1 = \frac{1}{r} OP \] i.e., \( OP_1 = 1 \) so that \( P_1 \) is reverse of \( P \) w.r.t. the unit circle with centre \( 0 \).

[The inverse of a point \( P \) w.r.t. a circle having centre \( 0 \) and radius \( k \) is defined as the point \( Q \) on \( OP \) such that \( OP \cdot OQ = k^2 \).]
The reflection $P'$ of $P$ in the real axis represents $w = -1/z$. Thus the transformation $w = 1/z$ is an inversion of $z$ w.r.t. the unit circle $|z| = 1$ followed by reflection of the inverse into the real axis.

Obviously, the transformation $w = 1/z$ maps the interior of the unit circle $|z| = 1$ into the exterior of the unit circle $|w| = 1$ and the exterior of $|z| = 1$ into the interior of $|w| = 1$.

However, the origin $z = 0$ is mapped to the point $w = -\infty$, called the point at infinity.

Note. This transformation $w = 1/z$ maps a circle in the $z$-plane to a circle in the $w$-plane or to a straight line if the circle in the $z$-plane passes through the origin.

The general equation of any circle in the $z$-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Let $w = u + iv = 1/z$, then $z = 1/w$ or $x + iy = 1/u + iv = u - iv/u^2 + v^2$ so that

$$x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{v}{u^2 + v^2}$$

Substituting the values of $x$ and $y$ in (1), we get

$$\frac{u^2}{(u^2 + v^2)^2} - \frac{v^2}{(u^2 + v^2)^2} + \frac{2gu}{(u^2 + v^2)^2} - \frac{2fv}{(u^2 + v^2)^2} + c = 0$$

or

$$\frac{u^2 + v^2}{(u^2 + v^2)^2} - \frac{2gu}{(u^2 + v^2)^2} + \frac{2fv}{(u^2 + v^2)^2} + c = 0$$

or

$$\frac{1}{u^2 + v^2} + \frac{2gu}{u^2 + v^2} + \frac{2fv}{u^2 + v^2} + c = 0$$

or

$$c(u^2 + v^2) + 2gu + 2fv + 1 = 0$$

If $c = 0$, the circle (1) does not pass through the origin and equation (2) represents a circle in the $w$-plane.

If $c = 0$, the circle (1) passes through the origin and equation (2) reduces to $2gu + 2fv + 1 = 0$ which is a straight line in the $w$-plane.

Regarding a straight line as a circle of infinite radius, we can say that the transformation $w = 1/z$ maps circles into circles.

### 4. Bilinear Transformation:  
(P.T.U., May 2005)

A transformation of the form $w = \frac{az + b}{cz + d}$ is called a bilinear or Mobius transformation.

The transformation given by (1) is conformal, since

$$\frac{du}{dz} = \frac{ad - bc}{(cz + d)^2} \neq 0$$

The inverse mapping of (1) is $z = -\frac{dw + b}{cw + a}$ which is also a bilinear transformation.

The transformation (1) can be written as

$$w = \frac{az + b}{cz + d} = az + wd - az - b = 0$$

which is linear both in $w$ and $z$ and hence the name bilinear transformation.

From (1), we observe that each point in the $z$-plane except the point $z = -d/c$ maps into a unique point in the $w$-plane. Similarly, from (2), we observe that each point in the $w$-plane except the point $w = \frac{a}{c}$ maps into a unique point in the $z$-plane. Considering the two exceptional points as points at infinity in respective planes, we can say that there is one to one correspondence between all points in the two plane.

Every bilinear transformation $w = \frac{az + b}{cz + d}$, $ad - bc \neq 0$ is the combination of basic transformations:

(i) translation: $w = z + c$

(ii) rotation and magnification: $w = cz$

(iii) inversion: $w = \frac{1}{z}$

By actual division, we have $w = \frac{a}{c} \cdot \frac{bc - ad}{c^2} - \frac{1}{z - \frac{d}{c}}$

Taking $w_1 = z + \frac{d}{c}$, $w_2 = \frac{1}{w_1}$, $w_3 = \frac{bc - ad}{c^2} - \frac{1}{w_2}$, we get $w = \frac{a}{c} + w_3$

Thus, by these transformations, we successively pass from $z$-plane to $w_1$-plane, from $w_1$-plane to $w_2$-plane, from $w_2$-plane to $w_3$-plane and finally from $w_3$-plane to $w$-plane.

Since each of these auxiliary transformations maps circles into circles, hence a bilinear transformation also maps circles into circles.

**Note 1. Cross-Section:** If four complex numbers $z_1$, $z_2$, $z_3$, $z_4$, are taken in order then

$$(z_1 - z_2)(z_3 - z_4)$$

is called the cross-section of $z_1$, $z_2$, $z_3$, $z_4$.

**Note 2. The cross-ratio is invariant under a bilinear transformation.**

Thus if $w_1$, $w_2$, $w_3$ and $w_4$ are the respective images of four distinct points $z_1$, $z_2$, $z_3$ and $z_4$, then

$$(w_1 - w_2)(w_3 - w_4) = (z_1 - z_2)(z_3 - z_4)$$

$$(w_1 - w_3)(w_2 - w_4) = (z_1 - z_3)(z_2 - z_4)$$

**Note 3.** In the bilinear transformation $w = \frac{az + b}{cz + d}$, $ad - bc \neq 0$ dividing the numerator and denominator of the right hand side by one of the four constants, we observe that there are only three independent constants. Hence three independent conditions are required to determine a bilinear transformation.
ILLUSTRATIVE EXAMPLES

Example 1. What is the region of $w$-plane into which the rectangular region is the $z$-plane bounded by the lines $x = 0$, $y = 0$, $x = 1$, $y = 2$ is mapped under the transformation, $w = z + (2 - i)$?

**Sol.** The given transformation is

$$w = z + (2 - i), \quad i.e., u + iv = x + iy + 2 - i$$

$$\therefore \quad u = x + 2 \quad \quad v = y - 1$$

$x = 0$ maps into $u = 2$

$y = 0$ maps into $v = -1$

$x = 1$ maps into $u = 3$

$y = 2$ maps into $v = 1$

So the mapped region PQRS is also a rectangle bounded by $u = 2$, $v = -1$, $u = 3$, $v = 1$ (shown in the figure below).

Example 2. Consider the transformation $w = e^z$ and determine the region in $w$-plane corresponding to the triangular region bounded by the lines $x = 0, y = 0$ and $x + y = 1$ in the $z$-plane.

**Sol.** The given transformation is

$$w = e^z = \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) (x + iy)$$

$$i.e., \quad u + iv = \frac{1}{\sqrt{2}}(1 + i)(x + iy) = \frac{1}{\sqrt{2}}[(x - y) + i(x + y)]$$

$$\therefore \quad u = \frac{1}{\sqrt{2}}(x - y) \quad v = \frac{1}{\sqrt{2}}(x + y)$$

$x = 0$ maps into $u = \frac{1}{\sqrt{2}}(-y), v = \frac{1}{\sqrt{2}}y$; $\therefore \quad v = -u$

$y = 0$ maps into $u = \frac{1}{\sqrt{2}}x, v = \frac{1}{\sqrt{2}}x$; $\therefore \quad v = u$

$x + y = 1$ maps into $u = \frac{1}{\sqrt{2}}, \quad v = \frac{1}{\sqrt{2}}$


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Example 3. Under the transformation $w = \frac{1}{z}$, find the image of the following curves:

(i) $|z - 2i| = 2.$

(ii) $y - x + 1 = 0.$

**Sol.** The given transformation is

$$w = \frac{1}{z} \quad \text{or} \quad z = \frac{1}{w} \quad \text{or} \quad x + iy = \frac{1}{w + iv} = \frac{u - iv}{u^2 + v^2}$$

so that

$$x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = -\frac{v}{u^2 + v^2} \quad \ldots (1)$$

(i) The given curve is $|z - 2i| = 2$ or $|x + iy - 2i| = 2$

or $x^2 + (y - 2)^2 = 2$ or $x^2 + y^2 - 4y = 0$ \ldots (2)

which is a circle in the $z$-plane with centre $(0, 2)$ and radius 2.

Substituting the values of $x$ and $y$ from (1) in (2), we get

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{4v^2}{u^2 + v^2} = 0 \quad \text{or} \quad \frac{u^2 + v^2}{(u^2 + v^2)^2} + \frac{4v^2}{u^2 + v^2} = 0$$

or

$$\frac{1}{u^2 + v^2} + \frac{4v^2}{u^2 + v^2} = 0$$

or

$$1 + 4v^2 = 0, \quad \text{a straight line which is the required image of the given curve.}$$

(ii) $y - x + 1 = 0$ maps into

$$-\frac{v}{u^2 + v^2} + \frac{u}{u^2 + v^2} + 1 = 0 \quad i.e., -u - v + u^2 + v^2 = 0$$

or

$$u^2 + v^2 - u - v = 0 \quad \text{which is a circle with centre at} \left( \frac{1}{2}, \frac{1}{2} \right) \quad \text{and radius} \quad \frac{1}{\sqrt{2}}$$

Example 4. Find the image of infinite strip $\frac{1}{4} \leq y \leq \frac{1}{2}$ under the transformation $w = \frac{1}{z}$.

Also show the regions graphically.
Sol. The given transformation is
\[ w = \frac{2z + 3}{z - 4} \]
so that
\[ x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{-v}{u^2 + v^2} \]
If \( y = \frac{1}{2} \), then \( \frac{u^2 + v^2}{2} = \frac{1}{2} \), or \( u^2 + v^2 + 2v = 0 \) or \( u^2 + (v + 1)^2 = 0 \).

Hence the infinite strip \( \frac{1}{4} \leq y \leq \frac{1}{2} \) is transformed into the region between the two circles
\[ u^2 + (v + 2)^2 = 4, \text{centre} (0, -2), \text{radius} 2 \]
and
\[ u^2 + (v + 1)^2 = 1, \text{centre} (0, -1), \text{radius} 1. \]

Example 5. Show that the image of the hyperbola \( x^2 - y^2 = 1 \) under the transformation
\[ w = \frac{1}{z} \]

is the Lemniscate \( \rho^2 = \cos 2\theta \).

(Bombay 2005; J.N.T.U. 2005)

Sol. Given transformation is \( w = \frac{1}{z} \). Let \( z = re^{i\phi} \); \( r e^{i\phi} = \frac{1}{r e^{-i\phi}} \).
\[ 1 = r e^{-i\phi} \implies r = 1 \quad \text{and} \quad \phi = -\phi = 0 \]

Equation of the hyperbola is
\[ x^2 - y^2 = 1 \quad \text{i.e.,} \quad r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1 \]
or
\[ r^2 \cos 2\theta = 1 \]
or
\[ \frac{1}{\rho^2} \cos 2(-\phi) = 1 \quad \text{or} \quad \rho^2 = \cos 2\phi. \]

Example 6. Show that the transformation \( w = \frac{2z + 3}{z - 4} \) maps the circle \( x^2 + y^2 - 4x = 0 \) onto the straight line \( 4u + 3 = 0 \).

Sol. The given transformation is
\[ w = \frac{2z + 3}{z - 4} \]
The inverse transformation is
\[ z = \frac{4w + 3}{w - 2} \]
Now the equation \( x^2 + y^2 - 4x = 0 \) can be written as \( z\bar{z} - 2(z + \bar{z}) = 0 \).
Substituting for \( z \) and \( \bar{z} \) from (1), we get
\[ \frac{4w^2 + 3}{w^2 - 2} + \frac{4w + 3}{w^2 - 2} = 0 \]
or
\[ 16w^2 + 12w + 9 = 2(4w + 3)w - 2(4w + 3)w - 2 = 0 \]
or
\[ (w + 3) + 33 = 0 \quad \text{or} \quad 22(2w) + 33 = 0 \quad \text{or} \quad 4u + 3 = 0. \]

Example 7. Show that \( w = \frac{z - i}{z + i} \) maps the real axis of the \( z \)-plane into the circle \( |w| = 1 \) and the half plane \( y > 0 \) into the interior of the unit circle \( |w| = 1 \) in the \( w \)-plane.

(P.T.U. May 2007)
Example 9. Determine the region of the \( w \)-plane into which the first quadrant of \( z \)-plane is mapped by the transformation \( w = z^2 \).

Sol. Proceeding as in example 5, \( \psi = 20 \)

For the first quadrant in \( z \)-plane, \( 0 < \theta < \frac{\pi}{2} \Rightarrow 0 < \phi < \pi \)

Hence the first quadrant of \( z \)-plane is mapped into the upper half of \( w \)-plane.

Example 10. Determine the region of the \( w \)-plane into which the region \( \frac{1}{2} \leq x \leq 1 \) and \( \frac{1}{2} \leq y \leq 1 \) is mapped by the transformation \( w = z^2 \).

Sol. The given transformation is \( w = z^2 \) or \( u + iv = (x + iy)^2 = (x^2 - y^2) + 2ixy \) so that

\[ u = x^2 - y^2 \quad \text{and} \quad v = 2xy \]

When \( x = \frac{1}{4} \), we have

\[ u = \frac{1}{4} - y^2 \quad \text{and} \quad v = y \quad \text{so that} \quad v^2 = \left( u - \frac{1}{4} \right) \]

(a left-handed parabola with vertex \( (\frac{1}{4}, 0) \) and latus rectum 1)

When \( x = 1 \), we have \( u = 1 - y^2 \) and \( v = 2y \) so that \( v^2 = -4(u - 1) \)

(a left-handed parabola with vertex \( (1, 0) \) and latus rectum 4)

When \( y = \frac{1}{2} \), we have \( u = x^2 - \frac{1}{4} \) and \( v = x \) so that \( v^2 = u + \frac{1}{4} \)

(a right-handed parabola with vertex \( (-\frac{1}{4}, 0) \) and latus rectum 1)

When \( y = 1 \), we have \( u = x^2 - 1 \) and \( v = 2x \) so that \( v^2 = 4(u + 1) \)

(a right-handed parabola with vertex \( (-1, 0) \) and latus rectum 4)

Thus the rectangular region bounded by the lines \( x = \frac{1}{2}, x = 1 \) and \( y = \frac{1}{2}, y = 1 \) maps into the region bounded by the parabolas

\[ v^2 = (u - \frac{1}{4}), \quad v^2 = -4(u - 1) \quad \text{and} \quad v^2 = u + \frac{1}{4}, \quad v^2 = 4(u + 1). \]

Example 11. Show that the transformation \( w = z + \frac{1}{z} \) maps the circle \( |z| = c \) into the ellipse \( u = \left( \frac{c + 1}{c} \right) \cos \theta, \quad v = \left( c - \frac{1}{c} \right) \sin \theta \). Discuss the case when \( c = 1 \).

Sol. A point on the circle \( |z| = c \) can be written as \( z = \cos \theta + i \sin \theta \)

\[ w = z + \frac{1}{z} \quad \text{becomes} \quad w = \cos \theta + i \sin \theta + \frac{1}{\cos \theta + i \sin \theta} \]

or

\[ u + iv = \cos \theta + \frac{1}{\cos \theta} \quad \text{and} \quad v = \frac{1}{\cos \theta} \sin \theta \]

(a semicircle of radius \( \frac{1}{2} \) with center at \( (0, 0) \) on \( x \)-axis)

which are the parametric equations of an ellipse.

When \( c = 1 \), we have \( u = 2 \cos \theta \) and \( v = 0 \)

Since \( | \cos \theta | \leq 1 \), we get \(-2 \leq u \leq 2 \) and \( v = 0 \)

\( \psi \). The transformation gives a segment of the \( u \)-axis of length 4.

Example 12. Show that the transformation \( w = z + \frac{1}{z} \) converts the straight line \( \arg z = \alpha \)

\( \left( |z| < \frac{\pi}{2} \right) \) into a branch of the hyperbola of eccentricity see \( \alpha \).

Sol. Let \( z = re^{i\theta} \) and \( w = u + iv \)

\[ u + iv = re^{i\theta} + \frac{1}{r} e^{-i\theta} = r \cos \theta + i \sin \theta + \frac{1}{r} \cos \theta - i \sin \theta \]

\[ = \left( r + \frac{1}{r} \right) \cos \theta + i \left( r - \frac{1}{r} \right) \sin \theta \]

\[ u = \left( r + \frac{1}{r} \right) \cos \theta, \quad v = \left( r - \frac{1}{r} \right) \sin \theta \]

\( \arg z = \alpha \Rightarrow \theta = \alpha \)

\[ u = \left( r + \frac{1}{r} \right) \cos \alpha, \quad v = \left( r - \frac{1}{r} \right) \sin \alpha \]

Eliminate \( r \):

\[ r = \frac{u}{\cos \alpha}, r = \frac{v}{\sin \alpha} \]

\[ \left( \frac{u}{\cos \alpha} + \frac{v}{\sin \alpha} \right)^2 - \left( \frac{u}{\cos \alpha} - \frac{v}{\sin \alpha} \right)^2 = \frac{u^2}{\cos^2 \alpha} - \frac{v^2}{\sin^2 \alpha} \]

or

\[ \frac{u^2}{4 \cos^2 \alpha} - \frac{v^2}{4 \sin^2 \alpha} = 1 \quad \text{where} \quad \alpha < \frac{\pi}{2} \quad \text{:. this equation represents a branch of hyperbola in \( w \)-plane.} \]

Here \( a^2 = 4 \cos^2 \alpha, b^2 = 4 \sin^2 \alpha \).
Eccentricity $e$ of the hyperbola is given by
\[ b^2 = a^2 (e^2 - 1) \]
\[ 4 \sin^2 \alpha = 4 \cos^2 \alpha (e^2 - 1) \]
\[ e^2 = 1 + \tan^2 \alpha \]
\[ e = \sec \alpha \]
\[ \therefore \text{Eccentricity of the hyperbola} = \sec \alpha \]

**Example 13.** Discuss the transformation $w = e^z$ and show that it transforms the region between the real axis and a line parallel to real axis at $y = \pi$ into the upper half of the $w$-plane.

**Sol.** Let $w = Re^{\phi}$, then the given transformation becomes
\[ Re^{i\phi} = e^{x+iy} = e^x \cos y + ie^x \sin y \]
\[ R = e^x \; \text{and} \; \phi = y \]
The real axis i.e., $y = 0$ maps into the positive $u$-axis $\phi = 0$ in the $w$-plane.
The line $y = \pi$ maps into the negative $u$-axis $\phi = \pi$ in the $w$-plane.
Thus the region between the lines $y = 0$ and $y = \pi$ maps into the upper half of $w$-plane.

**Note.** The region between the lines $y = 0$ and $y = -\pi$ maps into the lower half of $w$-plane.
The region between the lines $y = c$ and $y = c + 2\pi$ maps into the whole of the $w$-plane, since $e^z$ is periodic with period $2\pi$.
The imaginary axis $x = 0$ maps into a unit circle $R = e^0 = 1$ in the $w$-plane.

**Example 14.** Show that the transformation $w = e^z$ is always conformal. Under the mapping, find the images of the regions.

(i) the line segment $0 < y < A, A < 2\pi, x < 0$

(ii) the rectangle bounded by the lines $x = 0, y = 0, x = 1$ and $y = \pi$

(iii) The rectangular region bounded by the lines $a \leq x \leq b, c \leq y \leq d$. (P.T.U., Dec. 2005)

**Sol.**
\[ f(z) = e^z \]
\[ \therefore \text{The transformation is conformable in the region of } w \text{ i.e., for all values of } z \]
Let $z = x + iy$ and $w = Re^{i\phi}$
\[ R = e^x, \; \phi = y \]

(i) For $0 < y < A, A < 2\pi, x < 0$ we have $0 < \phi < A$ i.e., $0 < \arg w < A < 2\pi$

(ii) For $0 < y < A, A < 2\pi, x < 0$ we have $0 < \phi < A$ i.e., $0 < \arg w < A < 2\pi$

The image curves are given in Fig. (i).

**FUNCTIONS OF A COMPLEX VARIABLE**

- Image is the interior of the portion of the circle $|R| = 1$ whose angle of rotation is $A < 2\pi$.
  
  - (ii) The rectangle formed by $x = 0, y = 0, x = 1$ and $y = \pi$
    \[ x = 0 \Rightarrow R = 1 \]
    \[ y = 0 \Rightarrow \phi = 0 \]
    \[ x = 1 \Rightarrow R = e \]
    \[ y = \pi \Rightarrow \phi = \pi \]

  - If we take $w = u + iv$, then $u + iv = e^{x+iy} = e^x (\cos y + i \sin y)$
    \[ u = e^x \cos y, \; v = e^x \sin y \]
    \[ x = 0, y = 0 \Rightarrow u = 1, v = 0 \]
    \[ x = 1, y = 0 \Rightarrow u = e, v = 0 \]
    \[ x = 1, y = \pi \Rightarrow u = e, v = 0 \]
    \[ x = 0, y = \pi \Rightarrow u = 1, v = 0 \]

  - The image curves are given in Fig. (ii).

- Image is the region included between two semicircles $|R| = 1$ and $|R| = e$

  - (iii) The rectangle bounded by the lines $a \leq x \leq b, c \leq y \leq d$
    \[ a \leq x \leq b \Rightarrow e^a \leq e^x \leq e^b \]
    \[ c \leq y \leq d \Rightarrow c \leq \phi \leq d \]
    
    - The image is shown in Fig. (iii)

The image is the region PQRS.
Example 15. Discuss the transformation \( w = \sin z \).

Sol. The given transformation is
\[
  w = x + iy = \sin (x + iy)
\]
or
\[
  u + iv = \sin x \cosh y + i \cos x \sinh y
\]
so that
\[
  u = \sin x \cosh y
\]
and
\[
  v = \cos x \sinh y
\]
Eliminating \( y \) from equation (1), we get
\[
  \frac{u^2}{\sin^2 x} - \frac{v^2}{\cosh^2 x} = 1
\]
Thus the straight lines \( x = c \) in the \( z \)-plane are mapped into confocal hyperbolae in the \( w \)-plane.

Eliminating \( x \) from equation (1), we get
\[
  \frac{u^2}{\cosh^2 y} - \frac{v^2}{\sinh^2 y} = 1
\]
Thus the straight lines \( y = c \) in the \( z \)-plane are mapped into confocal ellipses.

The lines \( x = 0 \) and \( y = 0 \) map into the lines \( u = 0 \) and \( v = 0 \) respectively in the \( w \)-plane.

Example 16. Show that transformation \( w = \cosh z \) maps the lines parallel to \( x \)-axis and lines parallel to \( y \)-axis into confocal central conics.

Sol. Given transformation is \( w = \cosh z \)
\[
  u + iv = \cosh (x + iy) = \cosh x \cos y + i \sinh x \sin y
\]
\[
  u = \cosh x \cos y
\]
\[
  v = \sinh x \sin y
\]
Lines parallel to \( x \)-axis are given by \( y = k \)
\[
  u = \cosh x \cos k
\]
\[
  v = \sinh x \sin k
\]
\[
  \cosh x = \frac{u}{\cos k}
\]
\[
  \sinh x = \frac{v}{\sin k}
\]
Squaring and subtracting, we get
\[
  \frac{u^2}{\cos^2 k} - \frac{v^2}{\sin^2 k} = 1, \text{ which is a hyperbola in } w \text{-plane.}
\]
Here
\[
  a^2 = \cos^2 k, \quad b^2 = \sin^2 k
\]
\[
  \sinh k = \cos^2 k \left( e^2 - 1 \right) \text{ or } e^2 = 1 + \tan^2 \sec k = \sec^2 k
\]
\[
  e = \sec k
\]
Foci of (2) are \(( \pm \infty, 0) = (\pm \cos k, \sec k, 0) = (\pm 1, 0, 0)

Centre of (2) is \((0, 0, 0)\)

Now lines parallel to \( y \)-axis are \( x = k' \)
\[
  u = \cosh k' \cos y
\]
\[
  v = \sinh k' \sin y
\]

Functions of a complex variable

\[
  \frac{u}{\cosh k'} = \cos x
\]
\[
  \frac{v}{\sinh k'} = \sin x
\]

Squaring and adding
\[
  \frac{u^2}{\cosh^2 k'} + \frac{v^2}{\sinh^2 k'} = 1 \quad \text{which is an ellipse}
\]

Here \( a^2 = \cosh^2 k', \quad b^2 = \sinh^2 k' \)

Eccentricity of ellipse is given by \( e^2 = a^2 - b^2 \text{ i.e., } \sinh^2 k' = \cosh^2 k' \left( 1 - e^2 \right) \) or \( e^2 = 1 - \tan^2 \sec k' = \sec^2 k' \)
\[
  e = \sec k'
\]
\[
  \text{Foci of (3) are } (\pm \infty, 0) = (\pm \cosh k', \sec k', 0) = (\pm 1, 0, 0)
\]
Centre of (3) is \((0, 0, 0)\)

The centres and foci of both the conics are same.

Hence lines \( || \text{ } x \text{-axis and } y \text{-axis maps into confocal central conics.} \)

Example 17. The bilinear transformation \( w = \frac{az + b}{cz + d} \) transforms the circle \( \arg \frac{z - z_1}{z - z_2} = \text{constant} \) into similar circle of \( \arg \frac{w - w_1}{w - w_2} = \text{constant} \) where \( w_1, w_2 \) corresponds to \( z_1, z_2 \) respectively.

Sol. Here
\[
  w = \frac{az + b}{cz + d}
\]
\[
  w_1 = \frac{a z_1 + b}{c z_1 + d}, \quad w_2 = \frac{a z_2 + b}{c z_2 + d}
\]
\[
  w - w_1 = \frac{a z + b}{c z + d} - \frac{a z_1 + b}{c z_1 + d}
\]
\[
  w - w_2 = \frac{a z + b}{c z + d} - \frac{a z_2 + b}{c z_2 + d}
\]
\[
  \frac{w - w_1}{w - w_2} = \frac{z - z_1}{z - z_2}
\]
\[
  \arg \frac{w - w_1}{w - w_2} = \arg \mu \left( \frac{z - z_1}{z - z_2} \right)
\]
\[
  = \arg \mu + \arg \frac{z - z_1}{z - z_2}
\]
\[
  = \arg \mu = \lambda
\]
\[
  k \text{ (say) where } k \text{ is real because argument of a complex number is always real.} \]
Now, \( \text{arg} \frac{w - w_1}{w - w_2} = k \); represents a circle in \( w \)-plane passing through the points \( w_1 \) and \( w_2 \) corresponding to \( z_1 \) and \( z_2 \) in the \( z \)-plane.

**Example 18.** Find the bilinear transformation which maps the points \( z = 1, i, -1 \) into the points \( w = i, 0, -i \).

Hence find the image of \( |z| < 1 \). \( \text{(Bombay 2006)} \)

**Sol.** Let the required bilinear transformation be \( w = \frac{az + b}{cz + d} \) \( \ldots (1) \)

Substituting the corresponding values of \( w \) and \( z \) in (1), we get

\[
\begin{align*}
&i = \frac{a + b}{c + d} \quad 0 = \frac{ai + b}{ci + d} - i = \frac{-a + b}{-c + d} \\
&b + ia = 0 \quad b = -ia \\
&(-a + b) + i(-c + d) = 0 \quad \text{Adding (2) and (4),} \\
&2b - 2ic = 0 \quad c = \frac{b}{i} = -a \\
&\text{Subtracting (4) from (2),} \\
&2a - 2id = 0 \quad d = \frac{a}{i} = -ia \\
&\text{Substituting for} \ b, c, d \ \text{in (1), we get} \\
&w = \frac{ax - ia}{-ax - ia} \quad \text{or} \quad w = i \frac{-i}{i + z} \\
\end{align*}
\]

which is the required bilinear transformation.

Now, from (5),

\[
\begin{align*}
&z = \frac{1 - w}{1 + w} \\
&\text{or} \quad |z| < 1 \text{ is mapped into the region} \\
&\left| \frac{1 - w}{1 + w} \right| < 1 \quad \text{or} \quad \left| \frac{1 - w}{1 + w} \right| < 1 \\
&\text{or} \quad |1 - w| < 1 + |w| \\
&\text{or} \quad |1 - u - iv| < |1 + u + iv| \quad \text{or} \quad (1 - u)^2 + v^2 < (1 + u)^2 + v^2 \\
&u > 0.
\end{align*}
\]

Hence the interior of the circle \( x^2 + y^2 = 1 \) in the \( z \)-plane is mapped into the entire half of the \( w \)-plane to the right of the imaginary axis.
17. Show that the transformation \( w = z + \frac{a^2 - b^2}{4z} \) transforms the circle \(|z| = \frac{1}{2}(a + b)\) in the \(z\)-plane into an ellipse of semi-axes \(a, b\) in the \(w\)-plane.

18. Find the bilinear transformation which maps:
(a) the points \( z = 1, i, -1 \) onto the points \( w = i, 0, -i \).
(b) the points \( z = 1, i, -1 \) onto the points \( w = 0, 1, \infty \).
(c) the points \( z = 0, -i, 1 \) into the points \( w = i, 1, 0 \). Find the image of the line \( y = mx \) under this transformation.

19. Show that the condition for transformation \( w = \frac{az + b}{cz + d} \) to make the circle \(|w| = 1\) correspond to a straight line in the \(z\)-plane is \(|a| = |c|\).

20. Find the bilinear transformation which maps the points \( z = 1, i, -1 \) onto the points \( w = 2, i, -2 \) respectively. Also find the fixed points of the transformation.

**Answers**

1. \((u - 3)^2 + (v - 2)^2 = 4\)
2. A triangle with vertices \((4, -1), (7, 1)\) and \((7, -3)\).
3. Rectangular region bounded by the lines \( u = -4, v = u, u + v = 4, v - u = 6 \).
4. A triangular region bounded by the lines \( u = 0, v = 0, u + v = 2 \).
5. \(-1 < u < 1, v > 0\)
6. \(w + \frac{3}{16} = \frac{5}{18}\)
7. \(0, \frac{1}{2}, \frac{1}{2}\)
8. \(v = 0\)
9. \(y > 0\)
10. \((0, 0), (4, 0), (0, 8), (-4, 0)\); and isosceles triangle.
11. Negative real axis, positive real axis, \(v^2 = 4(1 - u)\), \(v^2 = 4(1 + u)\).
14. Region bounded by the parabolas \(v^2 = 4(1 + u), u^2 = 1 - 2v\).
15. \(\frac{5}{2}, \frac{3}{2}\) or \((0, -2)\).
16. \(w = \frac{z + i}{z - i}\)
18. \(w = \frac{d(1 - z)}{1 + z}\)
19. \(w = \frac{2i - 6z}{iz - 3}\); For fixed points \(w = z\) gives
   \(iz^2 + 3z - 2i = 0\) or \(z^2 + 3iz - 2 = 0\)
   or \((z + i)(z - 2i) = 0\) \(\therefore z = i, 2i\)
4.1. SERIES OF COMPLEX TERMS

A series of the form 
\[ a_1 + ib_1 + (a_2 + ib_2) + \ldots + (a_n + ib_n) + \ldots \]
where \(a_1, b_1, a_2, b_2, \ldots\) are real numbers, is called a series of complex terms and can be expressed as

\[ \sum_{n=1}^{\infty} a_n + i \sum_{n=1}^{\infty} b_n \]  \hspace{1cm} (1)

The series (1) is convergent if \(\Sigma a_n\) and \(\Sigma b_n\) both are convergent.

If the series \(\Sigma a_n\) and \(\Sigma b_n\) converge to \(a\) and \(b\) respectively, then the series (1) is said to converge to \(a + ib\).

The necessary (not sufficient) condition for convergence of series (1) is \( \lim_{n \to \infty} (a_n + ib_n) = 0 \).

The series (1) is said to be absolutely convergent if the series

\[ |a_1 + ib_1| + |a_2 + ib_2| + \ldots + |a_n + ib_n| + \ldots \]

is convergent.

Since \(|a_n| \leq |a_n + ib_n| \) and \(|b_n| \leq |a_n + ib_n| \), we conclude that an absolutely convergent series is convergent. The converse is not true.

If the series of functions \( u_1(z) + u_2(z) + \ldots + u_n(z) + \ldots = \sum_{n=1}^{\infty} u_n(z) \)  \hspace{1cm} (2)

converges to \(S(z)\) and \(S_n(z)\) is its \(n\)th partial sum, then the series (2) is said to be uniformly convergent in a region \(R\), if given any positive number \(\varepsilon\) there exists a positive number \(N\), depending on \(\varepsilon\) but not on \(z\), such that for every \(z\) in \(R\),

\[ |S(z) - S_n(z)| < \varepsilon \] for all \(n > N\).

A series of the form

\[ a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots + a_n(z - a)^n + \ldots \]  \hspace{1cm} (3)

is called a power series in \((z - a)\).

If the series (3) is convergent for all values of \(z\) satisfying \(|z - a| < R\), i.e., for all values of \(z\) lying inside the circle with centre at \(z = a\) and radius \(R\), then this circle is called the circle of convergence and \(R\) is called the radius of convergence.

The power series (3) is uniformly convergent in a region \(R\) if there exists a series of positive terms \(\Sigma M_n\) such that \(|a_n(z - a)^n| \leq M_n\) for all values of \(z\) in the region \(R\).

This is called Wierstrass M-test for uniform convergence of a power series.
4.2. TAYLOR'S SERIES

If \( f(z) \) is analytic inside a circle \( C \) with centre at \( a \), then for all \( z \) inside \( C \),

\[
f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \ldots
\]

Proof. Let \( z \) be any point inside the circle \( C \). Draw a circle \( C_1 \) with centre at \( a \) and radius smaller than that of \( C \) such that \( z \) is an interior point of \( C_1 \). Let \( w \) be any point on \( C_1 \), then

\[
|z-a| < |w-a| \quad \text{i.e.,} \quad \frac{|z-a|}{|w-a|} < 1
\]

Now,

\[
\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a} \left[ 1 - \frac{z-a}{w-a} \right]^{-1}
\]

Expanding R.H.S. by binomial theorem as

\[
\frac{1}{w-z} = \frac{1}{w-a} \left[ 1 + \frac{z-a}{w-a} + \frac{(z-a)^2}{w-a} + \ldots \right]^{n}
\]

This series converges uniformly since \( \frac{|z-a|}{|w-a|} < 1 \). Multiplying both sides of (1) by

\[
\frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-a} dw + \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-a} d\left( \frac{z-a}{w-a} \right) + \ldots
\]

Since \( f(w) \) is analytic on and inside \( C_1 \), by Cauchy's formula,

\[
\frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw = f(z)
\]

Also

\[
f^{(a)}(a) = \frac{1}{2\pi i} \frac{f(w)}{w-z} dw \quad \text{i.e.,} \quad \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-a} d\left( \frac{z-a}{w-a} \right) + \ldots
\]

From (2), we have

\[
f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \ldots \quad (3)
\]

which is the required Taylor's series for \( f(z) \) about \( z = a \).

Cor. 1. Putting \( z = a + h \) in (3), we get

\[
f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + \ldots
\]

4.3. LAURENT'S SERIES

If \( f(z) \) is analytic inside and on the boundary of the ring shaped region \( R \) bounded by two concentric circles \( C_1 \) and \( C_2 \) of radii \( r_1 \) and \( r_2 \) respectively having centre at \( a \), then for all \( z \) in \( R \),

\[
f(z) = a_0 + a_1 (z-a) + a_2 (z-a)^2 + \ldots + a_{-1} (z-a)^{-1} + a_{-2} (z-a)^{-2} + \ldots
\]

where

\[
a_n = \frac{1}{2\pi i} \oint_{C_n} \frac{f(w)}{(w-a)^{n+1}} dw \quad \text{and} \quad a_n = \frac{1}{2\pi i} \oint_{C_n} \frac{f(w)}{(w-a)^{n-1}} dw
\]

Proof. Let \( z \) be any point in the region \( R \), then by Cauchy's integral formula for double connected region, we have

\[
f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z} dw \quad (1)
\]

For the first integral in (1), \( w \) lies on \( C_1 \):

\[
|z-a| < |w-a| \quad \text{i.e.,} \quad \frac{|z-a|}{|w-a|} < 1
\]

Now

\[
\frac{1}{w-z} = \frac{1}{w-a} \left[ 1 + \frac{z-a}{w-a} + \frac{(z-a)^2}{w-a} + \ldots \right]^{n+1}
\]

Multiplying both sides by \( \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw \) and integrating term by term w.r.t. \( w \), along the circle \( C_1 \), we get

\[
\frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-a} dw + \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-a} \frac{d}{d(k)} \frac{(z-a)^k}{w-a} + \ldots
\]

\[
= a_0 + a_1 (z-a) + a_2 (z-a)^2 + \ldots
\]

\[
= \sum_{k=0}^{\infty} a_k (z-a)^k \quad (2)
\]

For the second integral in (1), \( w \) lies on \( C_2 \):

\[
|w-a| < |z-a| \quad \text{i.e.,} \quad \frac{|w-a|}{|z-a|} < 1
\]
Now \[
\frac{1}{w-z} = \frac{1}{(w-a) - (z-a)} = \frac{1}{z-a} \left( \frac{1}{w-a} \right)^{-1} \]
\[
= \frac{1}{z-a} \left[ 1 + \frac{w-a}{z-a} + \left( \frac{w-a}{z-a} \right)^2 + \ldots \right]
\]

Multiplying both sides by \( \frac{1}{2\pi i} \int_C f(w) \) and integrating term by term w.r.t. \( w \), along the circle \( C \), we get
\[
- \frac{1}{2\pi i} \int_C f(w) \frac{1}{w-z} dw = \frac{1}{z-a} \frac{1}{2\pi i} \int_C f(w) \frac{1}{w-z} dw + \frac{1}{(z-a)^2} \cdot \frac{1}{2\pi i} \int_C (w-a)^2 f(w) \frac{1}{w-z} dw + \ldots
\]
\[
= a_0 (z-a)^{-1} + a_2 (z-c)^{-2} + a_3 (z-a)^{-3} + \ldots
\]
\[
= \sum_{n=1}^{\infty} a_n (z-a)^{-n}
\]

Substituting from (2) and (3) in (1), we get
\[
f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_n (z-a)^{-n}
\]

Note 1. In the statement of Laurent's series, \( a_n = \frac{1}{2\pi i} \int_C f(w) \frac{1}{(w-a)^{n+1}} dw = \frac{f^n(a)}{n!} \) because \( f(z) \) is not given to be analytic inside \( C \).

Note 2. In case \( f(z) \) is analytic inside \( C \), then \( a_0 = 0 \) and \( a_n = \frac{1}{2\pi i} \int_C f(w) \frac{1}{(w-a)^{n+1}} dw = \frac{f^n(a)}{n!} \)

and Laurent's series reduces to Taylor's series.

Note 3. If \( C \) is any simple closed curve which lies in the ring-shaped region \( R \) and encloses the circle \( C \), then
\[
\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^n} dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}} dw
\]

Laurent's series can be written as
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,
\]
where \( a_n = \frac{1}{2\pi i} \int_C f(w) \frac{1}{(w-a)^{n+1}} dw \).

Note 4. The process of finding the coefficient \( a_n \) by complex integration is complicated. In practice, we expand the function \( f(z) \) by binomial theorem or by some other method to obtain Taylor's or Laurent's series.
\[
\begin{align*}
\therefore \quad f(z) &= \frac{1}{z-1} - \frac{1}{z} \left( \frac{2}{z} \right)^{-1} - \frac{1}{z} \left( \frac{1}{z} \right)^{-1} \\
&= \frac{1}{z} \left( 1 + \frac{2}{z} + \frac{4}{z^2} + \cdots \right) - \frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \right) \\
&= \frac{1}{z} \left( 1 + \frac{3}{z} + \frac{8}{z^2} + \cdots \right) \\
&= z^{-2} + 3z^{-3} + 7z^{-4} + \cdots \text{ which is a Laurent's series.}
\end{align*}
\]

(d) For \(0 < |z - 1| < 1\), we have
\[
f(z) = \frac{1}{z - 1} - \frac{1}{z} = -\left( z - 1 \right)^{-1} - \left( 1 - (z - 1) \right)^{-1} = -\left( z - 1 \right)^{-1} - \left( 1 + (z - 1)^2 + (z - 1)^3 + \cdots \right).
\]

**Example 9.** Find the series expansion of \(f(z) = \frac{z^2 - 1}{z^2 + 5z + 6}\) about \(z = 0\) in the region

(i) \(|z| < 2\) \quad (ii) \(2 < |z| < 3\).


**Sol. Here**
\[
f(z) = \frac{z^2 - 1}{z^2 + 5z + 6} = 1 + \frac{3}{z + 2} - \frac{8}{z + 3} \quad \text{(Partial Fractions)}
\]

(i) When \(|z| < 2\), we have \(\frac{z}{2} < 1\) and hence \(\left| \frac{z}{3} \right| < 1\).
\[
\therefore \quad f(z) = 1 + \frac{3}{z + 2} - \frac{8}{z + 3}
\]

\[
= 1 + \frac{3}{2} \left( 1 + \frac{z}{2} \right)^{-1} - \frac{8}{3} \left( 1 + \frac{z}{3} \right)^{-1}
\]

\[
= 1 + \frac{3}{2} \left[ 1 - \frac{z}{2} + \left( \frac{z}{2} \right)^2 - \left( \frac{z}{3} \right)^2 + \cdots \right] - \frac{8}{3} \left[ 1 - \frac{z}{3} + \left( \frac{z}{3} \right)^2 - \left( \frac{z}{3} \right)^3 + \cdots \right]
\]

\[
= 1 + \frac{3}{2} \sum_{n=0}^{\infty} (-1)^n \left( \frac{z}{2} \right)^n - \frac{8}{3} \sum_{n=0}^{\infty} (-1)^n \left( \frac{z}{3} \right)^n
\]

which is a Taylor's series.

(ii) When \(2 < |z| < 3\), we have \(\frac{2}{z} < 1\) and \(\left| \frac{z}{3} \right| < 1\).
\[
\therefore \quad f(z) = 1 + \frac{3}{z + 2} - \frac{8}{z + 3}
\]

\[
= 1 + \frac{3}{2} \left( 1 + \frac{2}{z} \right) - \frac{8}{3} \left( 1 + \frac{1}{z} \right)
\]

\[
= 1 + \frac{3}{2} \left( 1 + \frac{2}{z} \right)^{-1} - \frac{8}{3} \left( 1 + \frac{1}{z} \right)\]

which is a Laurent's series.

**Example 3.** Expand \(\frac{1}{z^2 + 1} (z^2 + 2)\) as a Laurent's series valid for

(i) \(0 < |z| < 1\) \quad (ii) \(1 < |z| < \sqrt{2}\) \quad (iii) \(|z| > \sqrt{2}\).


**Sol. Here**
\[
f(z) = \frac{1}{z^2 + 1} (z^2 + 2) = \frac{1}{z^2 + 1} \frac{1}{z^2 + 2} \quad \text{(Partial Fractions)}
\]

(i) When \(0 < |z| < 1\), we have \(z^2 < 1\) and hence \(z^2 < 2\) so that \(\frac{z^2}{2} < 1\).
\[
\therefore \quad f(z) = \frac{1}{1 + z^2} - \frac{1}{2 \left( 1 + z^2 \right)} = (1 + z^2)^{-1} - \frac{1}{2} \left( 1 + \frac{z^2}{2} \right)
\]

\[
= (1 - z^2 + z^4 - z^6 + \cdots) - \frac{1}{2} \left[ 1 - \frac{z^2}{2} + \left( \frac{z^2}{2} \right)^2 - \left( \frac{z^2}{2} \right)^3 + \cdots \right]
\]

\[
= \left( 1 - \frac{1}{2} \right)^{-1} - \left( 1 - \frac{1}{2} \right)^2 z^2 - \frac{1}{2} \left( 1 - \frac{1}{2} \right)^3 z^4 + \frac{1}{2} \left( 1 - \frac{1}{2} \right)^5 z^6 + \cdots
\]

\[
= \sum_{n=1}^{\infty} (-1)^{n-1} \left( 1 - \frac{1}{2^n} \right) z^{2n-2}
\]

(ii) When \(1 < |z| < \sqrt{2}\), we have \(z^2 > 1\) and \(z^2 < 2\) so that \(\frac{1}{z^2} < 1\) and \(\frac{z^2}{2} < 1\).
\[
\therefore \quad f(z) = \frac{1}{z^2 \left( 1 + \frac{1}{z^2} \right)} - \frac{1}{2 \left( 1 + \frac{z^2}{2} \right)} = \frac{1}{z^2} \left( 1 + \frac{1}{z^2} \right)^{-1} - \frac{1}{2} \left( 1 + \frac{z^2}{2} \right)^{-1}
\]

\[
= \frac{1}{z^2} \left[ 1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots \right] - \frac{1}{2} \left[ 1 - \frac{z^2}{2} + \left( \frac{z^2}{2} \right)^2 - \left( \frac{z^2}{2} \right)^3 + \cdots \right]
\]

\[
= \left[ 1 - \frac{1}{z^2} \right] - \left( \frac{1}{z^2} \right)^2 + \left( \frac{1}{z^4} \right)^2 + \cdots + \frac{1}{2} \left[ 1 - \frac{z^2}{2} + \left( \frac{z^2}{2} \right)^2 + \left( \frac{z^2}{2} \right)^3 + \cdots \right]
\]

\[
= \left[ 1 - \frac{1}{z^2} \right]^2 + \left( \frac{1}{z^2} \right)^3 - \left( \frac{1}{z^4} \right)^4 + \cdots - \frac{1}{2} \left[ 1 - \frac{z^2}{2} + \left( \frac{z^2}{2} \right)^2 + \left( \frac{z^2}{2} \right)^3 + \cdots \right]
\]

\[
= \frac{1}{z^2 - \left( \frac{1}{z^2} \right)^2 + \left( \frac{1}{z^4} \right)^3 - \left( \frac{1}{z^6} \right)^4 + \cdots}
\]
Example 6. Expand the function \( \frac{\sin z}{z-\pi} \) about \( z=\pi \).

Sol. Putting \( z-\pi = t \), we have
\[
\frac{\sin z}{z-\pi} = \frac{-\sin t}{t}
\]
\[
= -\frac{1}{t} \left( t^3 - \frac{t^5}{3!} + \frac{t^7}{5!} - \ldots \right) = -1 + \frac{t^2}{2} - \frac{t^4}{3!} + \ldots = -1 + \frac{(z-\pi)^2}{2} - \frac{(z-\pi)^4}{3!} + \ldots
\]

Example 7. Expand \( f(z) = \frac{z^2}{(z+1)(z+2)} \) about \( z=-2 \).

Sol. To expand \( f(z) \) about \( z=-2 \), i.e., in powers of \( z + 2 \), we put \( z + 2 = t \).
\[
f(z) = \frac{z}{(z+1)(z+2)} = \frac{t-2}{(t-1)(t-1)} = \frac{2-t}{t-1} \quad \text{for} \quad 0 < |t| < 1
\]
\[
= \frac{2}{t} - \frac{1}{t} (1 + t + t^2 + t^3 + \ldots)
\]
\[
= \frac{2}{t} + 1 + t + t^2 + \ldots
\]
\[
= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \ldots
\]
which is the required Laurent's series.

Example 8. Expand \( \frac{e^{z^2}}{(z-1)^3} \) about the singularity \( z=1 \) in Laurent's series.

Sol. To expand \( f(z) = \frac{e^{z^2}}{(z-1)^3} \) about \( z=1 \), i.e., in powers of \( z-1 \), we put \( z-1 = t \) or \( z=t+1 \).
\[
f(z) = e^{(t+1)^2} = e^t \cdot e^{2t}
\]
\[
= e^t \left[ 1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \ldots \right]
\]
\[
= e^t \left[ 1 + t^2 + 2t^3 + \frac{2t^4}{3!} + \frac{2t^5}{5!} + \ldots \right]
\]
\[
= e^t \left[ \frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{z-1} + \frac{4}{3} (z-1) + \ldots \right]
\]
Example 9. Find Taylor's expansion of

(i) \( \frac{1}{(z+i)^2} \) about the point \( z = -i \) 
(ii) \( \frac{2z^2 + 1}{z^2 + z} \) about the point \( z = i \)

(M.D.U. May 2011)

Sol. (i) To expand \( f(z) = \frac{1}{(z+i)^2} \) about \( z = -i \), i.e., in powers of \( z + i \), we put \( z + i = t \)

\[ f(z) = \frac{1}{(t-i)^2} = \frac{1}{[1+t(-i)]^2} = \frac{1}{[1+t(-i)]^2} \]

\[ = \frac{1}{(1-i)^2} \left( 1 + \frac{t}{1-i} \right)^2 \]

\[ = \frac{1}{(1-i)^2} \left( 1 + \frac{t}{1-i} \right) \]

\[ = \frac{1}{2i} \left[ 1 - \frac{2t}{1-i} + \frac{3t^2}{(1-i)^2} - \frac{4t^3}{(1-i)^3} + ... \right] \]

\[ = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(1-i)^{n+1}} t^n \]

(ii) To expand \( f(z) = \frac{2z^2 + 1}{z^2 + z} \) about \( z = i \), i.e., in powers of \( z-i \), we put \( z - i = t \)

\[ f(z) = 2(t+i) - 2 + \frac{1}{t+i+1} \]

Now, \( \frac{1}{t+i} = \frac{1}{i} \left( \frac{1-i}{1+i} \right)^{-1} \)

\[ = \frac{1}{i} \left( 1 - \frac{t}{i} \right) + \frac{1}{i} \left( \frac{t^2}{i^2} \right) + ... \]

\[ = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{i^n} \]

Also, \( \frac{1}{t+i} = \frac{1}{(1+i)(1-t/1+i)} = \frac{1}{1+i} \left( 1 + \frac{t}{1+i} \right) \)

\[ = \frac{1}{1+i} \left[ 1 - \frac{t}{1+i} + \frac{t^2}{(1+i)^2} - \frac{t^3}{(1+i)^3} + ... \right] \]

\[ = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{(1+i)^n} \]

Substituting in (1), we have

\[ f(z) = -2 + 2i + 2t + \sum_{n=1}^{\infty} (-1)^{n-1} \left[ \frac{1}{i^n} + \frac{1}{(1+i)^n} \right] z^{n-1} \]

\[ = -2(1-i) + 2(z-i) + \sum_{n=1}^{\infty} (-1)^{n-1} \left[ \frac{1}{i^n} + \frac{1}{(1+i)^n} \right] (z-i)^{n-1} \]

---

**EXERCISE 4.1**

Expand the following functions as a Taylor's series:

1. \( \log(1+z) \) about \( z = 0 \).
2. \( \sin z \) about \( z = \frac{\pi}{4} \).
3. \( \frac{z}{(z+1)(z+2)} \) about \( z = 2 \).

Expand the following functions in Laurent's series or Taylor's series:

4. \( \frac{1}{z-2} \), for \( |z| > 2 \).
5. \( \frac{1}{z^2 - 4z + 3} \), for \( 1 < |z| < 3 \).
6. \( \frac{1}{z(z-1)(z-2)} \), for \( |z| > 2 \).
7. \( \frac{1 - \cos z}{z^2} \), about \( z = 0 \).
8. \( \frac{e^z}{(z-1)^2} \), about \( z = 1 \).
9. \( \frac{1}{z(z-1)(z-2)} \), for \( |z-1| < 1 \).
10. \( \frac{1}{z(z^2 - 1)} \), for \( 0 < |z| < 2\pi \).
11. \[ \frac{z^2 - 1}{(z + 2)(z + 3)} \text{, when} \\
(i) \ |z| < 2 \] \[ \text{(ii) } 2 < |z| < 3 \] \[ \text{(iii) } |z| > 3 \]

12. \[ \frac{1}{z^2 + 1} \text{, when} \\
(i) \ |z| < 1 \] \[ \text{(ii) } 1 < |z| < 3 \] \[ \text{(iii) } |z| > 3 \]

13. \[ 2z - 2 \text{, when} \\
(i) \ 0 < |z - 1| < 1 \] \[ \text{(ii) } |z - 1| > 3 \]

14. \[ (z - 2)(z + 3) \] \[ (z + 1)(z + 4) \] \[ (iii) \ |z| > 4 \]

**Answers**

1. \[ z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots \]

2. \[ \frac{1}{\sqrt{2}} \left[ 1 + \left( z - \frac{\pi}{4} \right) - \frac{1}{2!} \left( z - \frac{\pi}{4} \right)^2 - \frac{1}{3!} \left( z - \frac{\pi}{4} \right)^3 + \ldots \right] \]

3. \[ \frac{1}{2} \left( \frac{1}{2} - \frac{1}{3} \right) \left( z - 2 \right) + \left( \frac{1}{2} - \frac{1}{3} \right) \left( z - 2 \right)^2 = -\sum \frac{(-1)^{n+1}}{2^{n+1} - 3^n} \] \[ (z - 2)^{n+1} \]

4. \[ \frac{1}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \ldots \]

5. \[ \frac{1}{6} \left( \sum \frac{z^n}{n} - \frac{1}{2} \sum \frac{z^n}{n} \right) \]

6. \[ z^3 + 3z^2 + 3z + 1 + \ldots + (2^n - 1)z^{n+1} + \ldots \]

7. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}z^{2n-1}}{(2n+1)!} \]

8. \[ e \left[ (z - 1)^2 + (z - 1)^3 + \frac{1}{2!} \right] + \frac{1}{3!} \left( z - 1 \right)^4 + \frac{1}{4!} \left( z - 1 \right)^5 + \ldots \]

9. \[ \frac{1}{z - 1} - \sum_{n=1}^{\infty} \left( z - 1 \right)^{2n-1} \]

10. \[ z^2 - \frac{1}{2} + \frac{z^2}{12} + \frac{z^2}{720} + \ldots \]

11. \[ \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{2^n} \frac{z^n}{3^n} \]

12. \[ \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{3^n} \]

13. \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \frac{z^n}{3^n} \]

14. \[ 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1 + 4^n)z^n}{4^n} \]

**RESIDUES**

4.4. SINGULAR POINTS

A point at which a function \( f(z) \) ceases to be analytic is called a singular point or singularity of \( f(z) \). For example, \( z = 2 \) is a singular point of \( f(z) = \frac{1}{z - 2} \).

4.5. ISOLATED SINGULAR POINT AND NON-ISOLATED SINGULAR POINT

A singular point \( z = a \) of a function \( f(z) \) is called an isolated singular point if there exists a small circle with centre \( a \) which contains no other singular point of \( f(z) \). Otherwise it is called non-isolated singular point.

For example; consider \( f(z) = \frac{\pi}{z^2 - 1} \).

The function \( f(z) \) is analytic everywhere except at \( z = 1, z = -1, 1 \) are the isolated singularities of \( f(z) \) as there are no other singularities of \( f(z) \) in the neighbourhood of \( z = 1 \) and \( 1 \).

Again consider \( f(z) = \cot \left( \frac{\pi}{z} \right) = \frac{1}{\pi} \tan \frac{\pi}{z} \)

It is not analytic at points where tan \( \frac{\pi}{z} = 0 \) i.e., \( \frac{\pi}{z} = n\pi \) or \( z = \frac{1}{n} \).
Thus $z = 1, \frac{1}{2}, \frac{1}{3}, \ldots$, 0 are the singularities of $f(z)$, all of which are isolated except $z = 0$.

In the neighbourhood of $z = 0$, there are infinities number of other singularities

$$z = \frac{1}{n} \text{ where } n \text{ is large}$$

$.\Rightarrow$ $z = 0$ is non-isolated singularity of the given function.

4.6. TYPES OF SINGULARITIES

Let $f(z)$ be analytic within a domain $D$ except at $z = a$ (an isolated singularity). Then with $z = a$ as centre we can draw two circles $C$ (with small radius) and $C'$ (with large radius) within $D$ so that $f(z)$ is analytic within the annulus between $C$ and $C'$. If $z$ is any point in the annulus then we can expand $f(z)$ as a Laurent's series about $z = a$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{-\infty} a_n (z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \left[a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots + a_{-m}(z-a)^{-m} + \ldots \right]$$

It gives rise to three types of singularities

(i) Removable Singularity

If in $f(z)$ the series has no negative power terms i.e., $a_{-1}, a_{-2}, a_{-3}, \ldots$ are all zero then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

In this case $z = a$ is called a removable singularity.

Note: This type of singularity can be made to disappear by re-defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$.

e.g., $f(z) = \frac{1}{z-a}$ has removable singularity at $z = a$ because

$$\frac{\sin (z-a)}{z-a} = \frac{1}{z-a} \left[ (z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} \ldots \right] = 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} \ldots$$

has no term containing negative powers of $(z-a)$. However, this singularity can be removed and the function can be made analytic by redefining the function as follows:

$$f(z) = \frac{\sin (z-a)}{z-a} \text{ when } z \neq a$$

$$= 1 \text{ when } z = a$$

(ii) Essential Singularity. If in (1) i.e., in the expansion of $f(z)$ the series with negative powers does not terminate i.e., has an infinite number of terms then $z = a$ is called an essential singularity of $f(z)$

$$f(z) = \sin \frac{1}{z-a} = \frac{1}{z-a} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} \ldots$$

$$= (z-a)^{-1} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} \ldots$$

has an infinite number of negative powers of $z-a$.

(iii) Pole

If in (1) i.e., in $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{-\infty} a_n (z-a)^{-n}$, the series with negative powers has a finite number of terms, say $m$, so that $a_{-m} = 0$ and all further coefficients i.e., $a_{m-1}, a_{m-2}, a_{m-3}, \ldots$ are zero, then $z = a$ is called a pole of order $m$ e.g., $f(z) = \frac{\sin (z-a)}{(z-a)^4}$ has a pole of order 3 at $z = a$.

$$f(z) = \frac{1}{(z-a)^4} \left[ (z-a) - \frac{1}{3!} (z-a)^3 + \frac{1}{5!} (z-a)^5 \ldots \right]$$

$$= \frac{1}{(z-a)^4} - \frac{1}{3!} \frac{1}{(z-a)^2} + \frac{1}{5!} \frac{1}{(z-a)^4} \ldots$$

$$= (z-a)^{-4} - \frac{1}{3!} (z-a)^{-2} + \frac{1}{5!} (z-a)^{-4} \ldots$$

It has a finite number of (only two) negative power terms and all other are zero

$.\Rightarrow$ $z = a$ is a pole of $f(z)$ of order 3, $(a_{-3} = 1 \neq 0 \text{ and } a_{-4} = a_{-5} = \ldots = 0)$

Note: Poles are always isolated singularities

4.7. HOW TO DETECT REMOVABLE SINGULAR POINTS

If $\lim_{z \to a} f(z)$ exists and is finite then $z = a$ is a removable singular point

4.8. RESIDUE AT A POLE

If $z = a$ is an isolated singularity of $f(z)$ then $f(z)$ can be expressed expanded in Laurents series about $z = a$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{-\infty} a_n (z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots + a_{-m}(z-a)^{-m} + \ldots$$

Then coefficient of $(z-a)^{-1}$ i.e., $a_{-1}$ is called the residue of $f(z)$ at $z = a$ and is written as

$$a_{-1} = \text{Res} \{ f(z) \}$$

Since $a_{-1} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$

$$\Rightarrow \oint_C f(z) dz = 2\pi i \text{ Res} \{ f(z), a \}$$
4.9. CAUCHY'S RESIDUE THEOREM

If \( f(z) \) is analytic at all points inside and on a simple closed curve \( C \), except at a finite number of isolated singular points within \( C \), then

\[
\oint_C f(z) \, dz = 2\pi i \text{(sum of residues at singular points within } C) \]

Proof. Around each of the isolated singular points \( a_1, a_2, \ldots, a_n \), draw small non-intersecting circles \( C_{a_1}, C_{a_2}, \ldots, C_{a_n} \) lying wholly inside \( C \), with centres at \( z = a_1, a_2, \ldots, a_n \) respectively.

Since \( f(z) \) is analytic in the multiply connected region bounded by \( C, C_{a_1}, C_{a_2}, \ldots, C_{a_n} \) we have by Cauchy's theorem for multiply connected region.

\[
\oint_C f(z) \, dz = \sum \oint_{C_{a_i}} f(z) \, dz = 2\pi i \sum_{a_i} \text{Res} \{f(z), a_i\}
\]

\[
= 2\pi i \sum \text{Res} \{f(z), a_i\} = 2\pi i \sum \text{Res} \{f(z), a_i\}
\]

4.10. CALCULATION OF RESIDUES

(i) If \( f(z) \) has a simple pole (i.e., pole of order 1) at \( z = a \), then \( \text{Res} \{f(z), a\} = \lim_{z \to a} (z - a) f(z) \).

Since \( z = a \) is a pole of order 1, the Laurent's series becomes

\[
f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots + a_n(z - a)^{-1}.
\]

Multiplying both sides by \( (z - a) \), we get

\[
\lim_{z \to a} (z - a) f(z) = a_1 = \text{Res} \{f(z), a\}.
\]

(ii) If \( f(z) \) has a pole of order \( m \) at \( z = a \), then

\[
\text{Res} \{f(z), a\} = \frac{1}{(m - 1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} [(z - a)^m f(z)]
\]

Since \( z = a \) is a pole of order \( m \), the Laurent's series becomes

\[
f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots + a_{-m} (z - a)^{-m}.
\]

Multiplying both sides by \( (z - a)^m \), we get

\[
(z - a)^m f(z) = a_0 (z - a)^m + a_1 (z - a)^{m+1} + a_2 (z - a)^{m+2} + \cdots + a_{-m} (z - a)^{-m}.
\]

Differentiating both sides \( (m - 1) \) times w.r.t. \( z \) and taking the limit as \( z \to a \), we get

\[
\lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} [(z - a)^m f(z)] = a_1 = \text{Res} \{f(z), a\}.
\]

(3) Another formula for \( \text{Res} \{f(z), a\} \)

Let \( f(z) = \frac{f(z)}{\psi(z)} \), where \( \psi(z) = (z - a) \psi(z), f(a) = 0 \)

Then \( \text{Res} \{f(z), a\} = \lim_{z \to a} (z - a) f(z) \)

\[
= \lim_{z \to a} \frac{(z - a) f(z)}{\psi(z)} = \lim_{z \to a} \frac{f(z)}{\psi(z)} \psi(a + (z - a))
\]

[Expanding \( \psi(z) \) and \( \psi(z) \) in ascending powers of \( (z - a) \)]

\[
= \lim_{z \to a} \frac{\phi(z) + (z - a) \phi'(z) + \cdots}{\psi(a) + (z - a) \psi'(a) + \cdots}
\]

\[
= \lim_{z \to a} \frac{(z - a) \psi'(a) + \cdots}{\psi(a) + (z - a) \psi'(a) + \cdots}
\]

\[
= \lim_{z \to a} \frac{(z - a) \psi'(a) + \cdots}{2!}
\]

ILLUSTRATIVE EXAMPLES

Example 1. Discuss the nature of singularities of the following functions:

(i) \( f(z) = \frac{z - \sin z}{z^2} \) at \( z = 0 \)

(ii) \( f(z) = e^{z^3} \) at \( z = a \)

(iii) \( f(z) = \frac{1}{\sin z - \cos z} \) at \( z = \frac{\pi}{4} \)

Sol. (i) \( f(z) = \frac{z - \sin z}{z^2} = \frac{1}{z^2} \left[ (z - \sin z) \frac{z^2}{z^2} \right]
\]

\[
= \frac{1}{3} - \frac{z^2}{5!} + \frac{z^4}{7!} + \cdots
\]
Since there are no negative power terms of \( z \), \( z = 0 \) is a removable singularity.

Aliter:  
\[
\lim_{z \to 0} \frac{z - \sin z}{z^3} = \lim_{z \to 0} \frac{-\cos z}{3z^2} = \lim_{z \to 0} \frac{\sin z}{6z} = 1 \\
[0 \text{ form, apply L'Hospital's rule}]
\]

Limit is finite \( z = 0 \) is a removable singularity

(ii)  
\[ f(z) = e^{\frac{1}{z-a}} = 1 + \frac{1}{z-a} + \frac{1}{2!} (z-a)^2 + \frac{1}{3!} (z-a)^3 + \ldots + \infty \]
\[ = 1 + (z-a)^{-1} + \frac{1}{2!} (z-a)^{-2} \frac{1}{3!} (z-a)^{-3} + \ldots + \infty \]

It has infinite number of negative power terms of \( z = a \); \( z = a \) is an essential singularity.

(iii) \[ f(z) = \frac{1}{\sin z - \cos z} \text{ at } z = \frac{\pi}{4} \]

Put \( z = \frac{\pi}{4} + t \)

\[ \phi(t) = \frac{1}{\sin \left( \frac{\pi}{4} + t \right) - \cos \left( \frac{\pi}{4} + t \right)} \]
\[ = \frac{1}{\sqrt{2} \sin t + \sqrt{2} \cos t} = \frac{1}{\sqrt{2} \left( t^2 + 2 \right)^{\frac{3}{2}} 5!^{\ldots} \infty} \]
\[ = \frac{1}{\sqrt{2} \left( 1 - \left( \frac{t^2 + 2}{5} \right)^{\frac{3}{2}} \right) \ldots} \]
\[ = \frac{1}{\sqrt{2} \left( 1 + \frac{t^2 + 2}{5} + \ldots + \infty \right) \ldots} \]
\[ = \frac{1}{\sqrt{2} \left( 1 + \frac{1}{t^2 + 2} \frac{1}{5} \right) \ldots} \]
\[ = \frac{1}{\sqrt{2} \left( 1 + \frac{1}{t} \frac{1}{3} \ldots + \infty \right) \ldots} \]

Since \( a_1 = \frac{\sqrt{2}}{2} \neq 0 \) and \( a_2 = a_3 = \ldots = 0 \) \( t = 0 \) is a simple pole of \( f(t) \). Hence \( z = \frac{\pi}{4} \) is a simple pole of \( f(z) \).

Example 2. What type of singularity have the following functions:

(i) \[ \frac{e^{2z}}{(z-1)^t} \]

(ii) \[ z = e^{z} \]

(iii) \[ \frac{1}{1-e^{z}} \]

(M.D.U. May 2011)
It has a finite number of negative power terms.

\[ a_{-1} = \frac{1}{1} = 0 \quad \text{and} \quad a_{-2} = a_{-3} = \ldots = 0 \]

Also \( 1-e^{z} = 0 \Rightarrow e^{z} = 1 = e^{2\pi i} \), \( n \) being an integer.

\[ z = 2n\pi (n = 0, \pm 1, \pm 2, \ldots) \]

\[ \frac{1}{1-e^{z}} \] has a simple pole at \( z = 0, \pm 2\pi i, \pm 4\pi i, \ldots \)

**Example 3.** Find the residue at \( z = 0 \) of

\[ f(z) = \frac{1 + e^{z}}{\sin z + z \cos z} \]

(i) \( f(z) = z \cos \frac{1}{z} \)  

(ii) \( f(z) = \csc^{2} z \)  

(iii) \( f(z) = \frac{1 + e^{z}}{\sin z + z \cos z} \)

**Sol.**

\[ f(z) = z \cos \frac{1}{z} = \frac{1}{z} \left( 1 - \frac{1}{2!} \frac{1}{z^2} + \frac{1}{4!} \frac{1}{z^4} \ldots \right) \]

\[ = \frac{1}{2} \left( z - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 \ldots \right) \ldots \]

\[ = \frac{1}{2} \left( z - 1 \ldots \right) \]

which is Laurent's expansion about \( z = 0 \)

\[ a_{-1} = \text{coeff. of } z^{-1} = \frac{1}{2} \]

By definition of residue (art 4.8)

Residue of \( f(z) \) at \( z = 0 \) is \(-\frac{1}{2}\)

(ii) \( f(z) = \csc^{2} z = \frac{1}{(\sin z)^2} = (\sin z)^{-2} \)

\[ = \left[ \frac{1}{2!} z^2 + \frac{1}{4!} z^4 \ldots \right]^{-2} = \frac{1}{z^2} \left[ 1 - \left( \frac{z^2}{3!} \frac{z^4}{5!} \ldots \right) \right]^{-2} \]

\[ = \frac{1}{z^2} \left[ 1 + 2 \left( \frac{z^2}{3!} \frac{z^4}{5!} \ldots \right) + 3 \left( \frac{z^4}{5!} \ldots \right) \ldots \right]^{-2} \]

\[ = \frac{1}{z^2} \left[ 1 + \frac{1}{3} z^2 + \frac{1}{15} z^4 \ldots \right]^{-2} \]

\[ = \frac{1}{z^2} \left( 1 + \frac{1}{3} z^2 + \frac{1}{15} z^4 \ldots \right) \]

Here \[ a_{-1} = \text{Coeff. of } z^{-1} = 0 \]

\[ \therefore \text{Res. of } f(z) \text{ at } z = 0 \text{ is } 0 \]

(iii) \( f(z) = \frac{1 + e^{z}}{\sin z + z \cos z} \)

\[ z = 0 \] is a simple pole of \( f(z) \)

\[ \therefore \text{Res. } f(z) = \lim_{z \to 0} \frac{1 + e^{z}}{z \cos z} = \lim_{z \to 0} \frac{1 + \frac{z}{z} e^{z}}{z \cos z} = \frac{1 + 1}{1} = \frac{2}{1} = 1 \]

**Example 4.** Determine the poles of the function \( f(z) = \frac{z^2}{(z-1)^2(z+2)} \) and the residue at each pole.

**Sol.** The function \( f(z) \) has a pole of order 2 at \( z = 1 \) and a simple pole at \( z = -2 \)

Residue of \( f(z) \) at \( z = 1 \) is

\[ \lim_{z \to 1} \frac{d}{dz} \left[ \frac{z^2}{(z-1)^2(z+2)} \right] = \lim_{z \to 1} \frac{dz}{dz} \left[ \frac{z^2}{(z+2)} \right] \]

\[ = \frac{z^2}{(z+2)^2} \cdot \frac{(z+2) - 2z}{z+2} \]

\[ = \frac{z^2 - 4z}{(z+2)^2} \]

\[ = \frac{5}{9} \]

Residue of \( f(z) \) at \( z = -2 \),

\[ \lim_{z \to -2} \left[ f(z) \right] = \lim_{z \to -2} \frac{z^2}{(z-1)^2(z+2)} = \frac{4}{9} \]

**Example 5.** Find the sum of the residues of the function \( f(z) = \frac{\sin z}{z \cos z} \) at its poles inside the circle \( |z| = 2 \).

**Sol.** The function \( f(z) \) has simple poles at \( z = 0, \pm \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \). Of these, \( z = 0, \pm \frac{\pi}{2} \) lie inside \( |z| = 2 \).

Residue of \( f(z) \) at \( z = 0 \) is \( \lim_{z \to 0} \frac{\sin z}{z \cos z} = 0 \)

Residue of \( f(z) \) at \( z = \frac{\pi}{2} \) is

\[ \lim_{z \to \frac{\pi}{2}} \left( z - \frac{\pi}{2} \right) f(z) = \lim_{z \to \frac{\pi}{2}} \frac{\sin z}{z \cos z} = \frac{0}{0} \]

\[ = \lim_{z \to \frac{\pi}{2}} \frac{\cos z + \sin z}{\cos z} \]

\[ = \frac{\frac{\pi}{2}}{2} \quad \text{[By L'Hospital's Rule]} \]

Similarly, residue of \( f(z) \) at \( z = -\frac{\pi}{2} \) is \( \frac{2}{\pi} \).
\[ z^2 + 2z + 5 = 0 \quad \text{i.e.,} \quad z = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i \]

(i) Both the poles lie outside the circle \( |z| = 1 \).

\[ \text{By Cauchy's theorem, we have} \int_C \frac{z-3}{z^2 + 2z + 5} \, dz = 0 \]

(ii) Only the pole \( z = -1 + 2i \) lies inside the circle \( |z + 1 - i| = 2 \) whose centre is at \(-1 + i\) and radius is 2.

Residue of \( f(z) \) at \( z = -1 + 2i \) is

\[ \lim_{z \to -1 + 2i} (z + 1 - 2i) f(z) = \lim_{z \to -1 + 2i} \frac{z^3 - 3}{z^2 + 2z + 5} = \]

\[ = \lim_{z \to -1 + 2i} \frac{(z + 1 - 2i)(z + 1 + 2i)}{(z + 1 + 2i)(z + 1 - 2i)} \]

\[ = \lim_{z \to -1 + 2i} \left( z + 1 + 2i \right) \frac{z - 3}{-1 + 2i - 1 + 2i} = \frac{2(i - 2)}{2i} = i - 2 \]

(iii) Only the pole \( z = -1 - 2i \) lies inside the circle \( |z + 1 + i| = 2 \), i.e., \( |z + 1 - i| = 2 \) whose centre is at \(-1 + i\) and radius 2.

Residue of \( f(z) \) at \( z = -1 - 2i \) is

\[ \lim_{z \to -1 - 2i} (z + 1 + 2i) f(z) = \lim_{z \to -1 - 2i} \frac{z^3 - 3}{z^2 + 2z + 5} = \]

\[ = \lim_{z \to -1 - 2i} \frac{(z + 1 + 2i)(z + 1 - 2i)}{(z + 1 + 2i)(z + 1 - 2i)} \]

\[ = \lim_{z \to -1 - 2i} \frac{z^3 - 3}{z^2 + 2z + 5} = \frac{2(-i - 2)}{2i} = -i - 2 \]

Example 8. Evaluate \( \oint_C \frac{z^2 - 1}{z(z + 1)(z - 3)} \, dz \), where \( C \) is the circle \( |z| = 2 \).

\[ \text{Sol. Here} \ f(z) = \frac{z^2 - 1}{z(z + 1)(z - 3)} \text{ has three simple poles at} \ z = 0, -1, 3 \text{ of which only } \]

\[ z = 0 \text{ lies inside the circle } \ |z| = 2 \]

Residue of \( f(z) \) at \( z = 0 \) is

\[ \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{2z - 1}{z(z + 1)(z - 3)} = \frac{1}{3} \]

Residue of \( f(z) \) at \( z = -1 \) is

\[ \lim_{z \to -1} (z + 1) f(z) = \lim_{z \to -1} \frac{2z - 1}{z(z + 1)(z - 3)} = -3 \]

By residue theorem,

\[ \oint_C \frac{2z - 1}{z(z + 1)(z - 3)} \, dz = 2\pi i \text{ (sum of residues)} \]

\[ = 2\pi i \left( \frac{1}{3} - \frac{3}{4} \right) = \frac{5\pi i}{6} \]

Example 9. Evaluate \( \oint_C \frac{e^z}{\cos \pi z} \), where \( C \) is the unit circle \( |z| = 1 \).

\[ \text{Sol. Here} \ f(z) = \frac{e^z}{\cos \pi z} \text{ has simple poles at} \ z = \pm 1, \pm \frac{3}{2}, \pm \frac{5}{2}, \ldots \text{ of which only} \ z = \pm \frac{1}{2} \]

lie inside the circle \( |z| = 1 \).
Residue of \( f(z) \) at \( z = \frac{1}{2} \) is

\[
\lim_{z \to \frac{1}{2}} (z - \frac{1}{2})f(z) = \lim_{z \to \frac{1}{2}} \frac{(z - \frac{1}{2})e^{e^z}}{\cos \pi z} = \lim_{z \to \frac{1}{2}} \frac{(z - \frac{1}{2})e^z + e^z}{-\pi \sin \pi z} = \frac{e^{\frac{1}{2}}}{-\pi}.
\]

By L'Hôpital's Rule.

Similarly, residue of \( f(z) \) at \( z = -\frac{1}{2} \) is \( \frac{e^{-\frac{1}{2}}}{\pi} \).

By residue theorem, \( \oint_C f(z) dz = 2\pi i \) (sum of residues).

\[
= 2\pi i \left( \frac{e^{\frac{1}{2}}}{\pi} - \frac{e^{-\frac{1}{2}}}{\pi} \right) = -4i \left( \frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{2} \right) = -4i \sinh \frac{1}{2}.
\]

**EXERCISE 4.2**

Determine the poles of the following functions and the residue at each pole:

1. \( \frac{2z + 1}{z^2 - z - 2} \)
2. \( \frac{z + 1}{z^2(z - 2)} \)
3. \( \frac{z^2}{(z - 1)(z - 2)} \)
4. \( \frac{1 - e^{2z}}{z^4} \)
5. \( \frac{e^z}{z^2 + n^2} \)
6. \( \frac{z}{\cos z} \)
7. \( \frac{z^2 - 2z}{(z + 1)(z^2 + 4)} \) (J.N.T.U. 2005)

Evaluate the following integrals:

8. \( \oint_C \frac{2z + 2 - 2}{z - 4} \) dz, where \( C \) is a closed curve containing the point \( z = 4 \) in its interior.
9. \( \oint_C \frac{1 - 2z}{2z - 1} \) dz, where \( C \) is the circle \( |z| = 1.5 \) (M.D.U. Dec. 2006)
10. \( \oint_C \frac{z}{z - 1} \) dz, where \( C \) is the circle \( |z - 2| = \frac{1}{2} \) (Madras 2006)
11. \( \oint_C \frac{13z - 7}{z^2(z - 2)} \) dz, where \( C \) is the circle \( |z| = 2 \).
12. \( \oint_C \frac{z - 2z^2 + \cos \pi z^2}{z(z - 2)} \) dz, where \( C \) is the circle \( |z| = 3 \). (U.P.T.U. 2005; V.T.U. 2006; M.D.U. Dec. 2009)
13. \( \oint_C \frac{1 - \cos 2z - 3z}{z^3 - 2z} \) dz, where \( C \) is the circle \( |z - 3| = 1 \).

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14. \( \oint_C \frac{z \sin z}{1 - z} dz \), where \( C \) is the circle \( |z| = 1.5 \).
15. \( \oint_C \frac{dz}{z^2 + 9} \), where \( C \) is the circle \( |z| = 5 \) (M.D.U. May 2009)
16. \( \oint_C \frac{z^2}{(z - 1)^2(z + 2)} dz \), where \( C \) is the circle \( |z| = \frac{5}{2} \).
17. \( \oint_C \frac{z^2 - 2z}{(z + 1)^2(z + 4)} dz \), where \( C \) is the circle \( |z| = 10 \) (U.P.T.U. 2009).
18. \( \oint_C \frac{z^3 - 2z}{(z - 1)(z + 2)(z - 3)} dz \), where \( C \) is the circle \( |z| = \frac{5}{2} \).
19. \( \oint_C \frac{dz}{\sinh z} \), where \( C \) is the circle \( |z| = 4 \).
20. \( \oint_C \frac{dz}{\tan z} \), where \( C \) is the circle \( |z| = 2 \).

**Answers**

1. \( z = -1, 2, \frac{1}{3}, \frac{5}{3} \)
2. \( z = 0, 2, -3, 3, \frac{3}{4}, \frac{3}{4} \)
3. \( z = 1, 2, 1, 0 \)
4. \( z = 0, -\frac{4}{3} \)
5. \( z = \pm n \pi i \pm \frac{i}{2\pi} \)
6. \( z = (2n + 1)\pi \pm (2n + 1)\pi \pm \frac{\pi}{2} \)

7. \( z = -1, 2\pi - \frac{7 + i}{25}, 25 \)
8. \( 44ni \)
9. \( 3ni \)
10. \( -2ni \)
11. \( 0 \)
12. \( 4ni(n + 1) \)
13. \( 1ni \)
14. \( 2ni \) sec \((1 + \tan 1) \)
15. \( -\frac{\pi}{3}, 0 \)
16. \( 2ni \)
17. \( 0 \)
18. \( -\frac{27ni}{8} \)
19. \( -2ni \)
20. \( -4ni \)

**4.11. APPLICATION OF RESIDUES TO EVALUATE REAL INTEGRALS**

The residue theorem provides a simple and elegant method for evaluating many important definite integrals of real variables. Some of these are illustrated below.

(1) Integrals of the type \( \int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta \), where \( F(\cos \theta, \sin \theta) \) is a rational function of \( \cos \theta \) and \( \sin \theta \).

Such integrals can be reduced to complex line integrals by the substitution \( z = e^{i\theta} \), so that
\[ dz = ie^{i\theta} \, d\theta, \ i.e., \ d\theta = \frac{dz}{iz} \]

Also
\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = 1 - \frac{1}{2} \left( z + \frac{1}{z} \right)
\]
\[
\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i} \left( z - \frac{1}{z} \right)
\]

As \( \theta \) varies from \( 0 \) to \( 2\pi \), \( z \) moves once round the unit circle in the anti-clockwise direction.

\[ \oint_{0}^{2\pi} F(\cos \theta, \sin \theta) \, d\theta = \oint_{C} \left( \frac{z + z^{-1}}{2} - \frac{1}{2i} \left( z - \frac{1}{z} \right) \right) \frac{dz}{iz} \]

where \( C \) is the unit circle \( |z| = 1 \).

The integral on the right can be evaluated by using the residue theorem.

### Illustrative Examples

**Example 1. Evaluate**
\[ \int_{0}^{2\pi} \frac{d\theta}{2 + \cos \theta} \]  
(P.T.U. 2006)

**Sol.** Put \( z = e^{i\theta} \) so that \( \cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right) \) and \( \, d\theta = \frac{dz}{iz} \).

Then
\[ \int_{0}^{2\pi} \frac{d\theta}{2 + \cos \theta} = \frac{1}{i} \int_{C} \frac{1}{2 \left( z + z^{-1} \right) i} \frac{dz}{iz} \]

where \( C \) is the circle \( |z| = 1 \).

The poles of the integrand are the roots of \( z^2 + 4z + 1 = 0 \), which are \( z = -2 \pm \sqrt{3} \).

Of the two poles, only \( z = -2 + \sqrt{3} \) lies inside the circle \( C \).

Residue at
\[ z = -2 + \sqrt{3} \]
\[ = \lim_{z \to -2 + \sqrt{3}} \frac{1}{z - \alpha} \frac{d}{dz} \left( \frac{1}{z^2 + 4z + 1} \right) \]
where \( \alpha = -2 - \sqrt{3} \quad \text{Form} \ 0 \)
\[ = \lim_{z \to -2 + \sqrt{3}} \frac{1}{2z + 4} = \frac{1}{2 \sqrt{3}} \]

\[ \Rightarrow \quad \text{By residue theorem,} \quad \frac{1}{i} \int_{C} \frac{dz}{z^2 + 4z + 1} = \frac{2\pi i}{2 \sqrt{3}} = \frac{\pi i}{\sqrt{3}} \]

Hence
\[ \int_{0}^{2\pi} \frac{d\theta}{2 + \cos \theta} = \frac{1}{i} \int_{C} \frac{dz}{z^2 + 4z + 1} = \frac{2\pi i}{\sqrt{3}} \]

**Example 2. Evaluate**
\[ \int_{0}^{2\pi} \frac{d\theta}{1 - 2a \sin \theta + a^2} \]  
\( 0 < a < 1 \)


**Sol.** Put \( z = e^{i\theta} \) so that \( \sin \theta = \frac{1}{2i} \left( z - \frac{1}{z} \right) \) and \( \, d\theta = \frac{dz}{iz} \).

Then
\[ \int_{0}^{2\pi} \frac{d\theta}{1 - 2a \sin \theta + a^2} = \frac{1}{i} \int_{C} \frac{1}{2 \left( z + z^{-1} \right) i} \frac{dz}{iz} = \frac{-1}{i} \int_{C} \frac{dz}{z^2 - (2a^2 + 1)z + a^2} \]

where \( C \) is the circle \( |z| = 1 \).

The integrand has simple poles at \( z = \frac{i}{a} \) and \( z = \frac{i}{a} \) of which only \( z = \frac{i}{a} \) lies inside the circle \( C \).

Residue at \( z = \frac{i}{a} \)
\[ = \lim_{z \to \frac{i}{a}} \frac{1}{z -\alpha} \frac{d}{dz} \left( \frac{1}{z^2 - (2a^2 + 1)z + a^2} \right) = \lim_{z \to \frac{i}{a}} \frac{1}{2z - 1} = \frac{-1}{i} \]

By residue theorem,
\[ \frac{1}{i} \int_{C} \frac{dz}{z - \frac{i}{a} + \frac{i}{a}} = \frac{2\pi i}{2a - 1} = \frac{2\pi}{2a - 1} \]

Hence
\[ \int_{0}^{2\pi} \frac{d\theta}{1 - 2a \sin \theta + a^2} = \frac{2\pi}{2a - 1} \]

**Example 3.** Prove that
\[ \int_{0}^{2\pi} \frac{\sin^2 \theta}{a + b \cos \theta} \, d\theta = \frac{2\pi (a - \sqrt{a^2 - b^2})}{b^2} \]
where \( 0 < b < a \).

**Sol.** Let
\[ I = \int_{0}^{2\pi} \frac{\sin^2 \theta}{a + b \cos \theta} \, d\theta = \int_{0}^{2\pi} \frac{1 - \cos 2\theta}{2a + 2b \cos \theta} \, d\theta \]

Real part of
\[ \int_{0}^{2\pi} \frac{1 - \cos 2\theta}{2a + 2b \cos \theta} \, d\theta \]

Put \( z = e^{i\theta} \) so that \( \cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right) \) and \( \, d\theta = \frac{dz}{iz} \).

Then
\[ \int_{0}^{2\pi} \frac{1 - \cos 2\theta}{2a + 2b \cos \theta} \, d\theta = \frac{1}{i} \int_{C} \frac{1 - \left( \frac{z - z^{-1}}{2} \right)^2}{2a + \frac{z + z^{-1}}{2} \left( z - \frac{1}{z} \right)} \frac{dz}{iz} \]

where \( C \) is the circle \( |z| = 1 \).

The poles of the integrand are the roots of \( bz^2 + 2az + b = 0 \), i.e.,
\[ z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = -a \pm \sqrt{a^2 - b^2} \]

Let
\[ \alpha = \frac{-a + \sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \beta = \frac{-a - \sqrt{a^2 - b^2}}{b} \]
Clearly, \( |\beta| > 1 \) so that \( z = \alpha \) is the only simple pole inside \( C \).

Also \( \frac{b}{i}z^2 + 2az + b = b(z - \alpha)(z - \beta) \).

Residue at \( z = \alpha \) is

\[
\lim_{z \to \alpha} \left( \frac{1 - z^2}{i(z - \alpha)(z - \beta)} \right) = \lim_{z \to \alpha} \left( \frac{1 - z^2}{ib(z - \alpha)} \right) = \frac{1 - \alpha^2}{ib(\alpha - \beta)} = \frac{1 - \alpha^2}{ib(\alpha - \beta)}.
\]

\[
= \frac{1 - \alpha^2}{ib(\alpha - \beta)} = \frac{\alpha(\alpha - \beta)}{ib(\alpha - \beta)} = \frac{\alpha - \beta}{ib}.
\]

\[
= \frac{\alpha - \beta}{ib} = \frac{\sqrt{a^2 - b^2}}{ib}.
\]

[\( \therefore \alpha \beta = 1 \)]

By residue theorem,

\[
\oint_C \frac{1 - z^2}{i(z - \alpha)(z - \beta)} dz = 2\pi i \left( \frac{1 - \alpha^2}{ib(\alpha - \beta)} \right) = 2\pi i \left( \frac{\alpha - \beta}{ib} \right) = 2\pi i \left( \frac{\sqrt{a^2 - b^2}}{ib} \right).
\]

Hence \( l = \text{Real part of} \left( \frac{1 - z^2}{i(z - \alpha)(z - \beta)} \right) dz = \frac{2\pi i}{b^2} \left( a - \sqrt{a^2 - b^2} \right) \).

Example 4. Evaluate \( \lim \frac{d\theta}{a + b \cos \theta} \) where \( a > 1 \).

Sol. Since \( \oint_C f(z) dz = 0 \), \( f(2a - x) = f(x) \)

\[
\oint_C f(z) dz = 2\pi i \left( \frac{f(x)}{x} \right) \]

[\( \therefore \cos (2\pi - \theta) = \cos \theta \)]

or

\[
\oint_C f(z) dz = \frac{2\pi i}{a + b \cos \theta} \]

Putting \( z = e^{i\theta} \), so that \( \cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right) \) and \( d\theta = \frac{dz}{iz} \), we have

\[
\oint_C f(z) dz = \frac{2\pi i}{a + b \cos \theta} = \frac{1}{iz} \oint_C \frac{f(z)}{a + b \cos \theta} dz = \frac{2\pi i}{a + b \cos \theta} \]

where \( C \) is the circle \( |z| = 1 \).

Proceeding as in example 3, residue at \( z = \alpha \) is

\[
\lim_{z \to \alpha} \left( \frac{1 - z^2}{i(z - \alpha)(z - \beta)} \right) = \lim_{z \to \alpha} \left( \frac{2}{ib(\alpha - \beta)} \right) = \frac{2}{ib}.
\]

\[
= \frac{b}{i} \cdot \frac{2}{2\sqrt{a^2 - b^2}} = \frac{1}{i\sqrt{a^2 - b^2}}.
\]

(2) Integrals of the type \( \int_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx \), where \( f(x) \) and \( F(x) \) are polynomials in \( x \) such that \( \frac{xf(x)}{F(x)} \rightarrow 0 \) as \( x \rightarrow \infty \) and \( F(x) \) has no zeros on the real axis.

Consider the integral \( \oint_C f(z) dz \)

over the closed contour \( C \) consisting of the real axis from \( -R \) to \( R \) and the semi-circle \( C_1 \) of radius \( R \) in the upper half plane.

We take \( R \) large enough so that all the poles of \( f(z) \) in the upper half plane lie within \( C \).

By residue theorem, we have

\[
\oint_C f(z) dz = 2\pi i \left( \text{sum of the residues of } f(z) \text{ in the upper half plane} \right)
\]

or

\[
\oint_C f(z) dz = \oint_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx = 2\pi i \left( \text{sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane} \right) \]

(\( \because \) on the real axis, \( z = x \))

If we put \( z = \Re^{i\theta} \) in the first integral on the left side, then \( R \) is constant on \( C_1 \) and as \( z \) moves along \( C_1 \), \( \theta \) varies from \( 0 \) to \( \pi \).

\[
\oint_{C_1} f(z) dz = \oint_{0}^{\pi} f(\Re^{i\theta}) \Re^{i\theta} i d\theta
\]

For large \( R \), \( \int_{0}^{\pi} f(\Re^{i\theta}) \Re^{i\theta} i d\theta \) is of the order \( \frac{R f(R)}{F(R)} \).

\[
\int_{0}^{\pi} f(\Re^{i\theta}) \Re^{i\theta} i d\theta \rightarrow 0 \text{ when } R \rightarrow \infty
\]

Hence from (1), we have

\[
\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx = 2\pi i \left( \sum \text{residues of } f(z) \text{ in the upper half plane} \right)
\]
ILLUSTRATIVE EXAMPLES

Example 1. Evaluate \( \int \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \, dx \) \( (a > 0, b > 0) \)

(P.T.U. 2007; Anna 2005, 2009)

Sol. Consider the integral \( \oint_C \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} \, dz \) over the closed contour \( C \) consisting of the real axis from \( -R \) to \( R \) and the semi-circle \( C_R \) of radius \( R \) in the upper half plane.

\[
\oint_C \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} \, dz = \int_{-R}^{R} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \, dx + \int_{C_R} \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} \, dz
\]

\( \quad \ldots (1) \)

(i) To evaluate \( \oint_C \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} \, dz \)

Poles of \( \phi(z) = \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} \) are given by \( z^2 + a^2 = 0, z^2 + b^2 = 0 \)

\[ i.e., \quad z = \pm ia, \pm ib \]

of these, only \( z = ia \) and \( z = ib \) lie in the upper half plane.

Residue of \( \phi(z) \) at \( z = ia \)

\[
= \lim_{z \to ia} \frac{z^2}{(z + ia)(z - ia)(z^2 + b^2)} = \frac{a}{2i(a^2 - b^2)}
\]

Residue of \( \phi(z) \) at \( z = ib \)

\[
= \lim_{z \to ib} \frac{z^2}{(z^2 + a^2)(z + ib)(z - ib)} = \frac{b}{2i(a^2 - b^2)}
\]

\( \therefore \) By Cauchy Residue Theorem

\[
\oint_C \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} \, dz = 2\pi i \text{ [Sum of residues of } \phi(z) \text{ within } C]\]

\[
= 2\pi i \left( \frac{a}{2i(a^2 - b^2)} - \frac{b}{2i(a^2 - b^2)} \right) = \pi \left( \frac{a - b}{a^2 - b^2} \right) = \frac{\pi}{a + b} \quad \ldots (2)
\]

(ii) Now \( \int_R^0 \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \, dx = -\int_{-R}^{0} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \, dx \)

\( \quad \ldots (3) \)

[:: Along x-axis, \( z = x \) and \( x \) varies from \( -R \) to \( R \) where \( R \to \infty \)]

(iii) Also \( \int_{C_R} \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} \, dz \)

Put \( z = Re^{i\theta} \) (any point on \( C_R \))

\[
= \int_0^\pi \frac{R^2 e^{i2\theta}}{(R^2 e^{2i\theta} + a^2)(R^2 e^{2i\theta} + b^2)} \, R e^{i\theta} \, i\theta \, d\theta
\]

\[
= \int_0^\pi \frac{R^3 e^{i3\theta}}{R^2 (a^2 + e^{2i\theta})(b^2 + e^{2i\theta})} \, d\theta \to 0 \text{ as } R \to \infty \]

\[
\therefore \oint_C \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} \, dz = 0 \quad \ldots (4)
\]

From (1), (2), (3) and (4)

\[
\frac{\pi}{a + b} = 0 + \int_{-R}^{0} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \, dx
\]

Hence \( \int_{-R}^{0} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \, dx = \frac{\pi}{a + b} \)

Example 2. Evaluate \( \int_0^\infty \frac{dx}{x^4 + 1} \)

(U.P.T.U. 2007)

Sol. \( \int_0^\infty \frac{dx}{x^4 + 1} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} \quad \ldots (1) \)

\[ \int_0^\infty f(x) \, dx = 2 \int_0^\infty f(x) \, dx \text{ if } f(x) \text{ is even} \]

\[ \int_0^\infty \frac{dx}{x^4 + 1} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} \quad \ldots (1) \]
Consider the integral \( \int_{C} \frac{dz}{z^{4} + 1} \) over the closed contour \( C \) consisting of the real axis from \(-R\) to \( R\) and the semi-circle \( C_{r} \) of radius \( R \) in the upper half plane.

\[ \int_{C} \frac{dz}{z^{4} + 1} = \int_{C_{r}} \frac{dz}{z^{4} + 1} + \int_{R} \frac{dz}{z^{4} + 1} \]

(\( i \)) To evaluate \( \int_{C_{r}} \frac{dz}{z^{4} + 1} \).

Poles of \( \phi(z) = \frac{1}{z^{4} + 1} \) are obtained by solving \( z^{4} + 1 = 0 \)

Now \( z^{4} + 1 = 0 \)

\[ z = (-1)^{\frac{1}{4}} = (\cos \pi + i \sin \pi)^{\frac{1}{4}} = [\cos (2m\pi + \pi) + i \sin (2m\pi + \pi)]^{\frac{1}{4}} \]

\[ = \cos \frac{(2m + 1)\pi}{4} + i \sin \frac{(2m + 1)\pi}{4} \] (By De Moivre's theorem)

where \( m = 0, 1, 2, 3 \)

When \( m = 0 \), \( z = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \)

When \( m = 1 \), \( z = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \)

When \( m = 2 \), \( z = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \)

When \( m = 3 \), \( z = \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} = -\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \)

Of these, only \( z = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} = e^{\frac{i\pi}{4}} \) and \( z = -\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} = e^{\frac{3i\pi}{4}} \)

lie in the upper half of \( z \)-plane.

Residue of \( \phi(z) \) at \( z = e^{\frac{i\pi}{4}} \) is

\[ \lim_{z \to e^{\frac{i\pi}{4}}} \frac{z - e^{\frac{i\pi}{4}}}{z^4 + 1} \]
\[ = \int_0^\infty \frac{R \cdot e^{ia \cdot dz}}{R^2 \cdot e^{ia} + 1} \] = \int_0^\infty \frac{e^{ia} \cdot dz}{R^2 \left(e^{ia} + \frac{1}{R^2}\right)} \rightarrow 0 \text{ as } R \rightarrow \infty
\]

\[ \int \frac{dz}{z^2 + 1} = 0 \]

From (2), (3), (4) and (5)

\[ \frac{\pi}{\sqrt{2}} = 0 + \int_{-\infty}^{\infty} \frac{dz}{z^2 + 1} \]

\[ \Rightarrow \frac{\pi}{\sqrt{2}} = 2 \int_0^\infty \frac{dx}{x^2 + 1} \]

[Using (1)]

\[ \Rightarrow \int_0^\infty \frac{dx}{x^2 + 1} = \frac{\pi}{2\sqrt{2}}. \]

The above method can also be applied to some cases, where \( f(x) \) contains trigonometric functions (sin ax or cos ax) i.e., integrals of the type \[ \int_{-\infty}^{\infty} \frac{\sin ax}{F(x)} \, dx \]

or \[ \int_{-\infty}^{\infty} \frac{\cos ax}{dx}, \text{ where } a \geq 0. \]

Example 3. Evaluate \[ \int_{0}^{\infty} \frac{\cos ax}{z^2 + 1} \, dx / (a \geq 0). \]

Sol.

\[ \int_{0}^{\infty} \frac{\cos ax}{z^2 + 1} \, dx = \text{Real part of} \int_{0}^{\infty} \frac{e^{iax}}{z^2 + 1} \, dx \]

Consider \[ \int_{C_1} \frac{e^{iax}}{z^2 + 1} \, dz \] where C consists of

(i) the semicircle \( C_1 \) of radius R in the upper half plane
(ii) the real axis from \(-R \) to R

\[ \int_{C_1} \frac{e^{iax}}{z^2 + 1} \, dz + \int_{-\infty}^{\infty} \frac{e^{iax}}{z^2 + 1} \, dz + \int_{\infty}^{-\infty} \frac{e^{iax}}{z^2 + 1} \, dz \]

(iii) To evaluate \[ \int_{0}^{\infty} \frac{e^{iax}}{z^2 + 1} \, dx \]

Poles of the integral are given by \( z^2 + 1 = 0 \) \( \Rightarrow z = \pm i \)

Only \( z = i \) lies within C

\[ \therefore \text{ Residue at } z = i \text{ (pole of order one)} = \frac{e^{ia} \cdot 0}{(z^2 + 1)} = \frac{e^{ia} \cdot 0}{2i} \]

\[ \Rightarrow \oint_{C} \frac{e^{ia}}{z^2 + 1} \, dz = 2\pi i \cdot \frac{e^{ia}}{2i} = \pi e^{-a} \]

(iii) To evaluate \[ \int_{0}^{\infty} \frac{e^{ia}}{z^2 + 1} \, dz \] Put \( z = Re^{i\theta} \)

Now

\[ \int_{0}^{\infty} \frac{e^{ia}}{z^2 + 1} \, dz = \frac{Re^{iaR \cos \theta}}{R^2 \cos^2 \theta + 1} = \frac{e^{ia} \cdot e^{iaR \cos \theta}}{R^2 \left(e^{2ia} + 1\right)} = \frac{R \cdot e^{iaR \cos \theta}}{R^2 + 1} \]

Since

\[ \left| e^{iaR \cos \theta} \right| = \left| \cos (aR \cos \theta) + i \sin (aR \cos \theta) \right| \leq 1 \]

\[ \left| e^{-aR \sin \theta} \right| \leq 1, \left| e^{ia} + \frac{1}{R^2} \right| \neq 0 \]

\[ \Rightarrow \int_{0}^{\infty} \frac{e^{ia}}{z^2 + 1} \, dz \rightarrow 0 \text{ as } R \rightarrow \infty \]

\[ \Rightarrow \int_{C} \frac{e^{ia}}{z^2 + 1} \, dz \rightarrow 0 \quad (3) \]

(iv) From (1), (2), (3), (4) \( \neq 0 \) \[ \Rightarrow \int_{0}^{\infty} \frac{e^{ia}}{z^2 + 1} \, dx = \int_{-\infty}^{\infty} \frac{e^{ia}}{z^2 + 1} \, dx \]

\[ \left| \begin{array}{c}
\int_{0}^{\infty} \frac{e^{ia}}{z^2 + 1} \, dx = \int_{-\infty}^{\infty} \frac{e^{ia}}{z^2 + 1} \, dx
\end{array} \right|
\]

\[ \therefore \text{ Along x-axis, } z = x \text{ and } x \text{ varies from } -R \text{ to } R \text{ where } R \rightarrow \infty \]

\[ \therefore \int_{-\infty}^{\infty} \frac{e^{ia}}{z^2 + 1} \, dx = 0 \]

\[ \Rightarrow \int_{0}^{\infty} \frac{e^{ia}}{z^2 + 1} \, dx = \frac{-\pi e^{-a}}{2} \]

\[ \Rightarrow \int_{0}^{\infty} \frac{e^{ia}}{z^2 + 1} \, dx = \frac{-\pi e^{-a}}{2} \]
EXERCISE 4.3

Evaluate the following integrals by contour integration:

1. \[ \int_{0}^{\pi} \frac{d\theta}{5 - 3 \cos \theta} \]
2. \[ \int_{0}^{\pi} \frac{d\theta}{5 + 3 \cos \theta} \]
3. \[ \int_{0}^{\pi} \frac{d\theta}{5 - 4 \cos \theta} \]
4. \[ \int_{0}^{\pi} \frac{d\theta}{5 + 4 \cos \theta} \]
5. \[ \int_{0}^{\pi} \frac{d\theta}{\cos \theta + r^2}, \quad (0 < r < 1) \]
6. \[ \int_{0}^{\pi} \frac{d\theta}{1 - 2 \cos \theta + a^2}, \quad (0 < a < 1) \]
7. \[ \int_{0}^{\pi} \frac{d\theta}{\sin \theta} \quad (a > b > 0) \]
8. \[ \int_{0}^{\pi} \frac{d\theta}{\cos \theta} \quad (a > b) \]
9. \[ \int_{0}^{\pi} \frac{d\theta}{\sin \theta} \quad (a > b) \]
10. \[ \int_{0}^{\pi} \frac{d\theta}{\cos \theta} \quad (a > b) \]
11. \[ \int_{0}^{\pi} \frac{d\theta}{\sin \theta} \quad (a > b) \]
12. \[ \int_{0}^{\pi} \frac{d\theta}{\cos \theta} \quad (a > b) \]
13. \[ \int_{0}^{\pi} \frac{d\theta}{\sin \theta} \quad (a > b) \]
14. \[ \int_{0}^{\pi} \frac{d\theta}{\cos \theta} \quad (a > b) \]
15. \[ \int_{0}^{\pi} \frac{d\theta}{\sin \theta} \quad (a > b) \]
16. \[ \int_{0}^{\pi} \frac{d\theta}{\cos \theta} \quad (a > b) \]
17. \[ \int_{0}^{\pi} \frac{d\theta}{\sin \theta} \quad (a > b) \]
18. \[ \int_{0}^{\pi} \frac{d\theta}{\cos \theta} \quad (a > b) \]
19. \[ \int_{0}^{\pi} \frac{d\theta}{\sin \theta} \quad (a > b) \]
20. \[ \int_{0}^{\pi} \frac{d\theta}{\cos \theta} \quad (a > b) \]
21. \[ \int_{0}^{\pi} \frac{d\theta}{\sin \theta} \quad (a > b) \]
22. \[ \int_{0}^{\pi} \frac{d\theta}{\cos \theta} \quad (a > b) \]
23. \[ \int_{0}^{\pi} \frac{d\theta}{\sin \theta} \quad (a > b) \]
24. \[ \int_{0}^{\pi} \frac{d\theta}{\cos \theta} \quad (a > b) \]
25. \[ \int_{0}^{\pi} \frac{d\theta}{\sin \theta} \quad (a > b) \]
CHAPTER 5

Probability Distributions

5.1. PROBABILITY

Here we define and explain certain terms which are used frequently.

(a) Trial and event. Let an experiment be repeated under essentially the same conditions and let it result in any one of the several possible outcomes. Then, the experiment is called a trial and the possible outcomes are known as events or cases.

For example: (i) Tossing of a coin is a trial and the turning up of head or tail is an event.

(ii) Throwing a die is a trial and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

(b) Exhaustive events. The total number of all possible outcomes in any trial is known as exhaustive events or exhaustive cases.

For example: (i) In tossing a coin, there are two exhaustive cases, head and tail.

(ii) In throwing a die, there are 6 exhaustive cases, for any one of the six faces may turn up.

(iii) In throwing two dice, the exhaustive cases are 6 × 6 = 36 for any of the 6 numbers from 1 to 6 on one die can be associated with any of the 6 numbers on the other die.

General, in throwing n dice, the exhaustive cases are 6^n.

(c) Favourable events or cases. The cases which entail the happening of an event are said to be favourable to the event. It is the total number of possible outcomes in which the specified event happens.

For example: (i) In throwing a die, the number of cases favourable to the appearance of a multiple of 3 are two viz. 3 and 6 while the number of cases favourable to the appearance of an even number are three, viz., 2, 4 and 6.

(ii) In a throw of two dice, the number of cases favourable to getting a sum 6 is 5, viz., (1, 5); (5, 1); (2, 4); (4, 2); (3, 3).

(d) Mutually exclusive events. Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes (i.e., rules out) the happening of all others. i.e., if no two or more than two of them can happen simultaneously in the same trial.

For example: (i) In tossing a coin, the events head and tail are mutually exclusive, since if the outcome is head, the possibility of getting tail in the same trial is ruled out.

(ii) In throwing a die, all the six faces numbered, 1, 2, 3, 4, 5, 6 are mutually exclusive since any outcome rules out the possibility of getting any other.

(e) Equally likely events. Events are said to be equally likely if there is no reason to expect any one in preference to any other.

5.2. MATHEMATICAL (or Classical) DEFINITION OF PROBABILITY

If a trial results in n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to the happening of an event E, then the probability of happening of E is given by

\[ p = \frac{m}{n} \]

Note 1. Since the number of cases favourable to happening of E is m and the exhaustive number of cases in n, therefore, the number of cases unfavourable to happening of E are n - m.

Note 2. The probability that the event E will not happen is given by

\[ q = \frac{n - m}{n} \]

Obviously, p and q are non-negative and cannot exceed unity, i.e., 0 ≤ p ≤ 1, 0 ≤ q ≤ 1.

Note 3. If P(E) = 1, then it is called a certain event i.e., the chance of its happening is certain per cent.

Note 4. If n cases are favourable to E and m cases are favourable to E (i.e., unfavourable to E), then exhaustive number of cases = n + m.

ILLUSTRATIVE EXAMPLES

Example 1. A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they will both be white.

Sol. Total number of balls = 7 + 6 + 5 = 18.

Out of 18 balls, 2 can be drawn in \( \binom{18}{2} \) ways.

\[ \therefore \text{ Exhaustive number of cases} = \binom{18}{2} = \frac{18 \times 17}{2 \times 1} = 153 \]

Out of 7 white balls, 2 can be drawn in \( \binom{7}{2} \) ways.

\[ \therefore \text{ Favourable number of cases} = \binom{7}{2} = \frac{7 \times 6}{2 \times 1} = 21 \text{ ways}. \]

Required probability

\[ = \frac{21}{153} = \frac{7}{51} \]
5.3. RANDOM EXPERIMENT

Events which can be repeated a number of times, essentially under the same conditions, and whose result cannot be predicted before hand are known as random experiments.

For example, rolling a dice, tossing a coin, taking out balls from an urn.

\textbf{Sample Space}. Out of the several possible outcomes of a random experiment, one and only one can take place in a trial. The set of all these possible outcomes is called the sample space for the particular experiment and is denoted by \( S \).

For example, if a coin is tossed, the possible outcomes are \( H \) (Head) and \( T \) (Tail).

Thus \( S = \{H, T\} \).

\[ S \subset \mathbb{C}_4 \]

\[ \begin{align*}
   \text{Exhaustive number of cases} & = \binom{52}{4} = \frac{52!}{4!48!} = 270725. \\
   \text{(i) There are 13 diamonds in the pack and 4 can be drawn out of them in } & \binom{13}{4} \text{ ways.} \\
   \text{Favourable number of cases} & = \binom{13}{4} = 715. \\
   \text{Required probability} & = \frac{715}{270725} = \frac{143}{54145} = \frac{11}{4165} \\
   \text{(ii) There are 4 suits, each containing 13 cards.} \\
   \text{Favourable number of cases} & = \binom{13}{4} \times \binom{13}{4} = 2197. \\
   \text{Required probability} & = \frac{2197}{270725} = \frac{20832}{20825} \\
   \text{(iii) 2 spades out of 13 can be drawn in } \binom{13}{2} \text{ ways.} \\
   \text{2 hearts out of 13 can be drawn in } \binom{13}{2} \text{ ways.} \\
   \text{Favourable number of cases} & = \binom{13}{2} \times \binom{13}{2} = 78 \times 78 = 6084. \\
   \text{Required probability} & = \frac{6084}{270725} = \frac{484}{20825} \\
\end{align*} \]

\[ \text{Example 3. A bag contains 50 tickets numbered } 1, 2, 3, \ldots, 50, \text{ of which five are drawn at random and arranged in ascending order of magnitude (} x_1 < x_2 < x_3 < x_4 < x_5 \text{). What is the probability that } x_2 = 30 ? \]

\[ \text{Sol. Exhaustive number of cases } = \binom{50}{5} \]

If \( x_2 = 30 \), then the two tickets with numbers \( x_1 \) and \( x_3 \) must come out of 29 tickets numbered 1 to 29 and this can be done in \( \binom{29}{2} \) ways. The other two tickets with numbers \( x_4 \) and \( x_5 \) must come out of the 20 tickets numbered 31 to 50 and this can be done in \( \binom{20}{2} \) ways.

\[ \begin{align*}
   \text{Favourable number of cases} & = \binom{29}{2} \times \binom{20}{2} = 351 \times 190 = 67140. \\
   \text{Required probability} & = \frac{67140}{\binom{50}{5}} = \frac{551}{15134} \\
\end{align*} \]

\[ \text{5.4. AXIOMS} \]

(i) With each event \( E \) (i.e., a sample point) is associated a real number between 0 and 1, called the probability of that event and is denoted by \( P(E) \). Thus \( 0 \leq P(E) \leq 1 \).

(ii) The sum of the probabilities of all simple (elementary) events constituting the sample space \( S \) is 1. Thus \( P(S) = 1 \).

(iii) The probability of a compound event (i.e., an event made up of two or more sample events) is the sum of the probabilities of the simple events comprising the compound event.

Thus, if there are \( n \) equally likely possible outcomes of a random experiment, then the sample space \( S \) contains \( n \) sample points and the probability associated with each sample point is \( \frac{1}{n} \) [By Axiom (ii)]

Now, if an event \( E \) consists of \( m \) sample points, then the probability of \( E \) is

\[ P(E) = \frac{1 + \frac{1}{n} + \cdots + \frac{1}{n}}{n} \text{ times } = \frac{n}{m} \]

Number of sample points in \( E \)

Number of sample points in \( S \).

This closely agrees with the classical definition of probability.

\[ \text{5.5. PROBABILITY OF THE IMPOSSIBLE EVENT IS ZERO, i.e., } P(\emptyset) = 0 \]

\[ \text{Proof. Impossible event contains no sample point. As such, the sample space } S \text{ and the impossible event } \emptyset \text{ are mutually exclusive.} \]

\[ \Rightarrow S \cup \emptyset = S \Rightarrow P(S \cup \emptyset) = P(S) \]

\[ \Rightarrow P(S) + P(\emptyset) = P(S) \Rightarrow P(\emptyset) = 0. \]

\[ \text{5.6. PROBABILITY OF THE COMPLEMENTARY EVENT } \bar{A} \text{ OR } A^c \text{ OF } A \text{ IS GIVEN BY} \]

\[ P(\bar{A}) = 1 - P(A) \]

\[ \text{Proof. } \bar{A} \text{ and } A \text{ are disjoint events. Also } A \cup \bar{A} = S \]

\[ \Rightarrow P(A \cup \bar{A}) = P(S) \]

\[ \Rightarrow P(A) + P(\bar{A}) = 1 \text{ Hence } P(\bar{A}) = 1 - P(A). \]
5.7. FOR ANY TWO EVENTS A AND B, \( P(A \cap B) = P(A) - P(B) \)

Proof. \( A \cap B = \{p : p \in B \text{ and } p \in A \} \)

Now, if \( A \cap B \) and \( A \cup B \) are disjoint sets and

\[ P(A \cap B) = P(A \cup B) - P(A) \]

\[ \therefore P(A \cap B) = P(A) - P(B) \]

Note. Similarly, it can be proved that \( P(A \cap B) = P(A) - P(B) \).

5.8. IF \( B \subset A \), THEN \((i) P(A \cap B) = P(A) - P(B) \) \((ii) P(B) \leq P(A) \)

Proof. When \( B \subset A \), and \( A \cap B \) is a disjoint set and their union is \( A \).

\[ B \cup (A \cap B) = A \]

\[ P(B \cup (A \cap B)) = P(A) \]

\[ P(B) + P(A \cap B) = P(A) \]

\[ P(A \cap B) = P(A) - P(B) \]

Now, if \( E \) is any event, then \( 0 \leq P(E) \leq 1 \), i.e., \( P(E) \geq 0 \)

\[ P(A \cap B) \geq 0 \Rightarrow P(A) - P(B) \geq 0 \]

[Using 1]

5.9. \( P(A \cap B) \leq P(A) \) and \( P(A \cap B) \leq P(B) \)

Proof. By 5.12 B \( \subset A \) \( \Rightarrow \) \( P(B) \leq P(A) \)

Since \( (A \cap B) \subset A \) and \( (A \cap B) \subset B \)

\[ P(A \cap B) \leq P(A) \text{ and } P(A \cap B) \leq P(B) \]

5.10. ADDITION THEOREM OF PROBABILITY

Statement. If \( A \) and \( B \) are any two events, then

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \cap B) \]

Proof. \( A \) and \( A \cap B \) are disjoint sets and there union is \( A \cup B \).

\[ A \cup B = A \cup (A \cap B) \]

\[ P(A \cup B) = P(A \cup (A \cap B)) = P(A) + P(A \cap B) \]

\[ = P(A) + P(A \cap B) \]

\[ = (A \cup B) \cup (A \cap B) = P(A \cup B) - P(A \cap B) \]

\[ \therefore \text{ if } A \cap B \text{ and } A \cup B \text{ are disjoint} \]

\[ (A \cup B) \cup (A \cap B) = B \]

\[ \therefore P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

Note 1. If \( A \) and \( B \) are two mutually disjoint events, then \( A \cap B = 0 \), so that \( P(A \cap B) = 0 \).

\[ \therefore P(A \cup B) = P(A) + P(B) \]

Note 2. \( P(A \cup B) \) is also written as \( P(A \cup B) \). Thus, for mutually disjoint events \( A \) and \( B \),

\[ P(A \cup B) = P(B) \]

\[ P(A \cap B) \text{ is also written as } P(AB) \]

5.11. IF \( A, B \) AND \( C \) ARE ANY THREE EVENTS, THEN

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \]

Or

\[ P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC) \]

Proof. Using the above Art. 5.10 for two events, we have

\[ P(A \cup B \cup C) = P(A \cup B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \]

By distributive law

\[ = [P(A) + P(B)] + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \]

\[ = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC) \]

or

\[ P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC) \]

5.12. IF \( A_1, A_2, ..., A_n \) ARE \( n \) MUTUALLY EXCLUSIVE EVENTS, THEN THE PROBABILITY OF THE HAPPENING OF ONE OF THEM IS

\[ P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1 + A_2 + ... + A_n) = P(A_1) + P(A_2) + ... + P(A_n) \]

Proof. Let \( N \) be the total number of mutually exclusive, exhaustive and equally likely cases in which \( m_1 \) are of which \( m_1 \) are of \( A_1 \), \( m_2 \) are of \( A_2 \), and so on.

Probability of occurrence of event \( A_1 \) \( = \frac{m_1}{N} \)

Probability of occurrence of event \( A_2 \) \( = \frac{m_2}{N} \)

Probability of occurrence of event \( \cdots \)

Probability of occurrence of event \( A_n \) \( = \frac{m_n}{N} \)

The events being mutually exclusive and equally likely, the number of cases favourable to the event

\[ A_1 \text{ or } A_2 \text{ or } \cdots \text{ or } A_n \text{ is } m_1 + m_2 + \cdots + m_n \]

\[ \therefore \text{ Probability of occurrence of one of the events } A_1, A_2, ..., A_n \text{ is } P(A_1 + A_2 + ... + A_n) \]

\[ = \frac{m_1 + m_2 + \cdots + m_n}{N} \]

\[ = \frac{m_1}{N} + \frac{m_2}{N} + \cdots + \frac{m_n}{N} \]

[Using (1)]

Note. The student should not get confused with Theorems 5.10, 5.11 and 5.12. Theorems 5.10 and 5.11 are for ANY events (mutually exclusive or not) whereas Theorem 5.11 is for mutually exclusive events.
ILLUSTRATIVE EXAMPLES

Example 1. In a given race, the odds in favour of four horses A, B, C, D are 1 : 3, 1 : 4, 1 : 5, 1 : 6 respectively. Assuming that a dead heat is impossible, find the chance that one of them wins the race.

Sol. Let \( p_1, p_2, p_3, p_4 \) be the probabilities of winning of the horses A, B, C, D respectively.

Since a dead heat (in which all the four horses cover same distance in same time) is not possible, the events are mutually exclusive.

Odds in favour of A are 1 : 3 \( \therefore p_1 = \frac{1}{1+3} = \frac{1}{4} \)

Similarly, 
\( p_2 = \frac{1}{5}, p_3 = \frac{1}{6}, p_4 = \frac{1}{7} \)

If \( p \) is the chance that one of them wins, then

\[ p = p_1 + p_2 + p_3 + p_4 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = 0.4206 \]

Example 2. A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?

Sol. Let 
\( A = \) the event of drawing a spade
\( B = \) the event of drawing an ace

and A and B are not mutually exclusive.

\( AB = \) the event of drawing the ace of spades

\[ P(A) = \frac{13}{52}, P(B) = \frac{4}{52}, P(A \cap B) = \frac{1}{52} \]

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \cap B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13} \]

5.13. CONDITIONAL PROBABILITY

The probability of the happening of an event \( E_1 \) when another event \( E_2 \) is known to have already happened is called Conditional Probability and is denoted by \( P(E_1|E_2) \).

Mutually Independent Events. An event \( E_1 \) is said to be independent of an event \( E_2 \) if

\[ P(E_1|E_2) = P(E_1) \]

i.e., if the probability of happening of \( E_1 \) is independent of the happening of \( E_2 \).

5.14. MULTIPLICATIVE LAW OF PROBABILITY (Or Theorem of Compound Probability)

The probability of simultaneous occurrence of two events is equal to the probability of one of the events multiplied by the conditional probability of the other, i.e., for two events A and B,

\[ P(A \cap B) = P(A) \times P(B|A) \]

where \( P(B|A) \) represents the conditional probability of occurrence of B when the event A has already happened.

ILLUSTRATIVE EXAMPLES

Example 1. A problem in mechanics is given to three students A, B, C whose chances of solving it are \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \) respectively. What is the probability that the problem will be solved?

Sol. The probabilities of A, B, C solving the problem are \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \) respectively.

The probabilities of A, B, C not solving the problem are \( 1 - \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4} \) i.e., \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \)

\[ \therefore \] The probability that the problem is not solved by any of them is \( \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{24} \)

Hence the probability that the problem is solved by at least one of them is \( 1 - \frac{1}{24} = \frac{23}{24} \)
Example 2. The odds that a book will be favourably reviewed by three independent critics are 5 to 2, 4 to 3 and 3 to 4 respectively. What is the probability that, of the three reviews, a majority will be favourable?

Sol. Let the three critics be A, B, C. The probabilities \( p_1, p_2, p_3 \) of the book being favourably reviewed by A, B, C are respectively.

\[
\begin{align*}
1 - \frac{5}{7} & = \frac{2}{7} \\
1 - \frac{4}{7} & = \frac{3}{7} \\
1 - \frac{3}{7} & = \frac{4}{7}
\end{align*}
\]

\( A \) majority will be favourable if the reviews of at least two are favourable.

(i) If A, B, C all review favourably, the probability is

\[
\frac{5 \times 4 \times 3}{7 \times 7 \times 7} = \frac{60}{343}
\]

(ii) If A, B review favourably and C reviews unfavourably, the probability is

\[
\frac{5 \times 4 \times 3}{7 \times 7 \times 7} \times \left(1 - \frac{3}{7}\right) = \frac{80}{343}
\]

(iii) If A, C review favourably and B reviews unfavourably, the probability is

\[
\frac{5 \times 3 \times 3}{7 \times 7 \times 7} \times \left(1 - \frac{4}{7}\right) = \frac{45}{343}
\]

(iv) If B, C review favourably and A reviews unfavourably, the probability is

\[
\frac{2 \times 4 \times 3}{7 \times 7 \times 7} \times \left(1 - \frac{5}{7}\right) = \frac{24}{343}
\]

Hence the probability that a majority will be favourable is

\[
\frac{60 + 80 + 45 + 24}{343} = \frac{209}{343}
\]

Example 3. A can hit a target 4 times in 5 shots; B 3 times 4 shots; C twice in 3 shots.

They fire a volley. What is the probability that at least two shots hit?

Sol. Probability of A's hitting the target = \( \frac{4}{5} \)

Probability of B's hitting the target = \( \frac{3}{4} \)

Probability of C's hitting the target = \( \frac{2}{3} \)

For at least two hits, we may have

(i) A, B, C all hit the target, the probability for which is

\[
\frac{4 \times 3 \times 3}{5 \times 4 \times 4} = \frac{24}{60} = \frac{1}{2}
\]

(ii) A, B hit the target and C misses it, the probability for which is

\[
\frac{4 \times 3 \times \left(1 - \frac{2}{3}\right)}{5 \times 4 \times 3} = \frac{3}{6} = \frac{1}{2}
\]

Example 4. A has 2 shares in a lottery in which there are 3 prizes and 5 blanks; B has 3 shares in a lottery in which there are 4 prizes and 6 blanks. Show that A's chance of success is to B's as 27 : 35.

Sol. A can draw two tickets (out of 3 + 5 = 8) in \( _8C_2 = 28 \) ways.

A will get all blanks in \( _8C_3 = 10 \) ways. :: A can win a prize in 28 - 10 = 18 ways

Hence A's chance of success = \( \frac{18}{28} = \frac{9}{14} \)

B can draw 3 tickets (out of 4 + 6 = 10) in \( _{10}C_3 = 120 \) ways; B will get all blanks in \( _{12}C_3 = 20 \) ways.

:: B can win a prize in 120 - 20 = 100 ways.

Hence B's chance of success = \( \frac{100}{120} = \frac{5}{6} \)

:: A's chance : B's chance = \( \frac{9}{14} : \frac{5}{6} = 27 : 35 \).

Example 5. A and B throw alternately with a single die, A having the first throw. The person who first throws ace is to win. What are their respective chances of winning?

Sol. The chance of throwing an ace with a single die = \( \frac{1}{6} \)

The chance of not throwing an ace with a single die = \( 1 - \frac{1}{6} = \frac{5}{6} \)

If A is to win, he should throw an ace in the first or third, fifth, ... throws.

If B is to win, he should throw an ace in the second or fourth or sixth, ... throws.

The chances that an ace is thrown in the first, second, third, ... throws are

\[
\frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \left(\frac{5}{6}\right)^2, \frac{1}{6}, \left(\frac{5}{6}\right)^3, \frac{1}{6}, ... \]

A's chance = \( \frac{1}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + ... \)

Sum of an infinite G.P. = \( \frac{a}{1-r} \)

B's chance = \( 1 - \frac{6}{11} = \frac{5}{11} \)
Example 6. Cards are dealt one by one from a well-shuffled pack until an ace appears. Show that the probability that exactly \( n \) cards are dealt before the first ace appears is

\[
\frac{d(51-n)(50-n)(49-n)}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47}.
\]

Sol. Let \( A \) be the event of drawing \( n \) non-ace cards and \( B \), the event of drawing an ace in the \((n+1)\)th draw.

Consider the event \( A \)

\( n \) cards can be drawn out of 52 cards in \( \binom{52}{n} \) ways.

\[
P(A) = \binom{48}{n-1} \binom{52}{n} = \frac{48!}{(48-n)!n!} \times \frac{52!}{52!} = \frac{48! \cdot (52-n)!}{(52-51-n)! \cdot 51 \cdot 50 \cdot \cdots \cdot (52-n)}.
\]

Consider the event \( B \)

\( n \) cards have already been drawn in the first \( n \) draws.

\[
P(B) = \frac{4}{52} = \binom{48}{n} \binom{51}{52} \binom{51}{n} = \frac{4}{52} \cdot \frac{51!}{(51-n)!} \cdot \frac{51!}{51} \cdot \frac{51!}{51-n}.
\]

Example 7. An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the latter. What is the probability that it is a white ball?

Sol. The two balls drawn from the first urn may be

(i) both white

(ii) both black

(iii) one white and one black.

Let these events be denoted by \( A, B, C \) respectively.

\[
P(A) = \binom{10}{2} = \frac{10 \times 9}{12} = \frac{45}{20} = \frac{9}{26},
\]

\[
P(B) = \frac{3 \times 2}{12} = \frac{1}{4},
\]

\[
P(C) = \frac{10 \times 3}{12} = \frac{10}{12} = \frac{5}{6}.
\]

When two balls are transferred from first urn to second urn, the second urn will contain

(i) 5 white and 5 black balls

(ii) 3 white and 7 black balls

(iii) 4 white and 6 black balls.

Let \( W \) denote the event of drawing a white ball from the second urn in the three cases (i), (ii) and (iii).

Now

\[
P(W/A) = \frac{5}{10}, \quad P(W/B) = \frac{3}{10}, \quad P(W/C) = \frac{4}{10}.
\]
10. Three newspapers A, B, C are published in a city and a survey of readers indicates the following: 20% read A, 14% read B, 16% read C. 8% read both A and B, 5% read both A and C, 4% read both B and C, 2% read all three. For a person chosen at random, find the probability that he reads none of the papers.

11. A problem in statistics is given to five students. Their chances of solving it are \(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\) and \(\frac{1}{6}\). What is the probability that the problem will be solved?

12. A can hit a target 3 times in 5 shots, B 2 times in 5 shots and C 3 times in 4 shots. They fire a volley. What is the probability that (i) both A and B hit, (ii) at least two should hit the target?

13. Three groups of children contain respectively 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys. One child is selected at random from each group. Show that the chances that the three selected consist of 1 girl and 2 boys are \(\frac{13}{22}\).

14. Suppose that 5 men out of 100 and 25 women out of 1000 are good orators. Assuming that there are equal numbers of men and women, find the probability of choosing a good orator.

15. A bag contains 10 balls, two of which are red, three blue and five black. Three balls are drawn at random from the bag. What is the probability that (i) the three balls are of different colours, (ii) two balls are of the same colour, (iii) all the balls are of the same colour.

16. It is 8:5 against a person who is 40 years old living till he is 70 and 4:3 against a person who is 50 living till he is 60. Find the probability that one at least of these persons will be alive 30 years hence.

17. An old car in a pharmacy shop is found to contain 60 capsules and 180 tablets. Half of capsules and tablets are of expiry date. Find the probability that an item picked at random from the caron is of expiry date or a capsule.

18. A has 3 shares in a lottery where there are 3 prizes and 6 blanks. B has one share in another, where there is just one prize and two blanks. Show that A has a better chance of winning a prize than B in the ratio 16:7.

19. A, B, and C, in order, toss a coin. The first one to throw a head wins. If A starts, find their respective chances of winning.

20. (a) A speaks truth in 75% cases and B in 80% cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

21. (a) Two persons toss an unbiased coin alternately on the understanding that the first who gets the head wins. If A starts the game, find their respective chances of winning.

22. (a) Two cards are randomly drawn from a deck of 52 cards and thrown away. What is the probability of drawing an ace in a single draw from the remaining 50 cards?

23. A committee consists of 9 students two of which are from 1st year, three from 2nd year and four from 3rd year. Three students are to be removed at random. What is the chance that (i) the three students belong to different classes, (ii) two belong to the same class and third to the different class, and (iii) the three belong to the same class?

24. Five men in a company of twenty are graduates. If 3 men and picked out of 20 at random, what is the probability that (i) they are all graduates, (ii) at least one is graduate?

25. If A, B, C are events such that \(P(A) = 0.3, P(B) = 0.4, P(C) = 0.6\), \(P(A \cap B) = 0.08, P(A \cap C) = 0.03, P(A \cap B \cap C) = 0.09\).

26. For two events A and B, let \(P(A) = 0.4, P(B) = 0.2\) and \(P(A \cup B) = 0.6\). Find \(P(A)\) if A and B are independent events.

27. A husband and wife appear in an interview for two vacancies in the same post. The probability of husband's selection is \(\frac{1}{3}\) and that of wife's selection is \(\frac{1}{5}\). What is the probability that (i) both of them will be selected, (ii) only one of them will be selected, and (iii) none of them will be selected.

28. Two dice are tossed once. Find the probability of getting an even number on the first die or a total of 8.

29. There are two bags. The first bag contains 5 white and 3 black balls and the second bag contains 3 white and 5 black balls. Two balls are drawn at random from the first bag and without noting their colours, put into the second bag. Then two balls are drawn from the second bag. Find the probability that the balls drawn are white and black.

30. A purse contains 2 silver and 4 copper coins. If a coin is picked at random from one of the two purses, what is the probability that it is a silver coin?

31. A man wants to marry a girl having qualities: white complexion—the probability of getting such a girl is one in twenty; good manners—most unfortunate 50% fell into both of these categories. What is the probability that a girl picked at random was able to see and hear satisfactorily?

32. The probabilities of A, B, C solving a problem are \(\frac{1}{3}, \frac{1}{2}, \frac{3}{7}\) respectively. If all the three try to solve the problem simultaneously, find the probability that only one of them will solve it.

33. A student takes an examination in four subjects \(a, b, c, d\). He estimates his chance of passing in them as \(a = \frac{4}{5}, b = \frac{3}{4}, c = \frac{5}{6}\) and \(d = \frac{2}{3}\). To qualify he must pass in at least two other subjects. What is the probability that he qualifies?

34. For any two events A and B, prove that \(P(A \cup B) \leq P(A) + P(B)\).
37. A student takes his examination in four subjects P, Q, R, S. He estimates his chances of passing in P as $\frac{3}{4}$, in Q as $\frac{5}{6}$, in R as $\frac{2}{3}$ and in S as $\frac{4}{5}$. To qualify, he must pass in P and at least two other subjects. What is the probability that he qualifies?

38. Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn at random. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?

39. A die is thrown three times. Events A and B are defined as below:

A: 4 appears on third throw
B: 6 and 5 appear respectively on first two throws

Find the probability of A given that B has already occurred.

40. Two dice are rolled. If E1 = (1, 3), F = (3, 2) and G = (2, 3, 4, 5, 6), find P(E ∩ F ∩ G) and P(E ∩ G).

41. A person has undertaken a construction job. The probabilities are 0.65 that there will be a strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.30 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.

### Answers

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**Quiz Section**

**Illustrative Examples**

**Example 1.** A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.

**Sol.** Let

- $E_1$: the ball is drawn from bag X
- $E_2$: the ball is drawn from bag Y
- $A$: the ball is red

We have to find $P(E_2|A)$. By Baye's Theorem,

$$P(E_2|A) = \frac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)}$$

Since the two bags are equally likely to be selected, $P(E_1) = P(E_2) = \frac{1}{2}$

Also $P(A|E_1) = P(A|E_2) = \frac{5}{9}$

Therefore, $P(E_2|A) = \frac{\frac{1}{2} \cdot \frac{5}{9}}{\frac{1}{2} \cdot \frac{5}{9} + \frac{1}{2} \cdot \frac{1}{9}} = \frac{5}{11}$
Example 2. In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% per cent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine B? (K.U.K. Dec. 2010; V.T.U. 2006)

Sol. Let \(E_1\), \(E_2\) and \(E_3\) denote the events that a bolt selected at random is manufactured by the machines A, B and C respectively and let \(H\) denote the event of its being defective. Then

\[
P(E_1) = 0.25, \quad P(E_2) = 0.35, \quad P(E_3) = 0.40
\]

The probability of drawing a defective bolt manufactured by machine A is

\[
P(H/E_1) = \frac{0.05}{0.25} = 0.20
\]

Similarly,

\[
P(H/E_2) = \frac{0.05}{0.35} = 0.04
\]

By Baye's Theorem, we have

\[
P(E_2/H) = \frac{P(E_2)P(H/E_2)}{P(E_1P(H/E_1) + P(E_2)P(H/E_2) + P(E_3)P(H/E_3))}
\]

\[
= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = \frac{0.0140}{0.0345} = 0.41
\]

Example 3. The contents of urns I, II and III are as follows:

- 1 white, 2 black and 3 red balls,
- 2 white, 1 black and 1 red balls, and
- 4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn. They happen to be white and red.

What is the probability that they come from urns \(I\), \(II\) or \(III\)? (K.U.K. 2006)

Sol. Let \(E_1\): urn I is chosen, \(E_2\): urn II is chosen, \(E_3\): urn III is chosen and \(A\): the two balls are white and red.

We have to find \(P(E_1/A), P(E_2/A)\) and \(P(E_3/A)\).

Now

\[
P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}
\]

\[
P(A/E_1) = P(\text{a white and a red ball are drawn from urn I}) = \frac{\binom{1}{1} \times \binom{3}{1}}{3 \times \binom{6}{2}} = \frac{1}{5}
\]

\[
P(A/E_2) = \frac{\binom{2}{1} \times \binom{1}{1}}{\binom{4}{2}} = \frac{1}{3}, \quad P(A/E_3) = \frac{\binom{4}{1} \times \binom{5}{1}}{\binom{12}{2}} = \frac{2}{11}
\]

By Baye's Theorem, we have

\[
P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}
\]

\[
= \frac{1 \times \frac{1}{5}}{1 \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{2}{11} \times \frac{2}{11}} = \frac{33}{118}
\]

Similarly, \(P(E_2/A) = \frac{55}{118}\) and \(P(E_3/A) = \frac{15}{59}\).

**EXERCISE 5.2**

1. (a) Two urns contain 4 white, 6 blue and 4 white, 5 blue balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is white, find the probability that it is drawn from the

(i) first urn

(ii) second urn.

(b) Of the cigarette smoking population, 70% are men and 30% women, 10% of these men and 20% of these women smoke 'WILLS'. What is the probability that a person seen smoking a 'WILLS' will be a man?

2. (a) Three urns contain 6 red, 4 black; 4 red, 6 black and 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the first urn.

(b) There are three bags:

- First containing 1 white, 2 red, 3 green balls;
- Second containing 2 white, 3 red, 1 green balls;
- Third containing 3 white, 1 red, 2 green balls.

Two balls are drawn from a bag chosen at random. They are found to be 1 red and 1 white. Find the probability that balls so drawn came from the second bag.

3. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2.5% of the items produced by machine A were defective and 1.5% produced by machine B were defective. If a defective item is drawn at random, what is the probability that it was produced by machine A?

4. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of accident is 0.01, 0.03 and 0.15 respectively. One of the insured persons meets an accident. What is the probability that he is a scooter driver?

5. A company has two plants to manufacture scooters. Plant I manufactures 70% of the scooter and plant II manufactures 30%. At plant I, 10% of the scooters are rated standard quality and at plant II, 50% of the scooters are rated standard quality. A scooter is chosen at random and is found to be of standard quality. What is the chance that it has come from plant II?

6. In a bolt factory, there are four machines A, B, C, D manufacturing 20%, 15%, 25% and 40% of the total output respectively. Of their outputs 5%, 4%, 3% and 2%, in the same order, are defective bolts. A bolt is chosen at random from the factory's production and is found defective. What is the probability that the bolt was manufactured by machine A or machine B?

7. A doctor is to visit a patient. From the past experience, it is known that the probability that he will come by train, bus, scooter or by other means of transport are respectively \(\frac{3}{10}, \frac{1}{5}, \frac{3}{10}\) and \(\frac{2}{5}\). The probabilities that he will be late are \(\frac{1}{4}, \frac{1}{5}\) and \(\frac{1}{12}\), if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

8. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.
9. A survey was conducted to find the supplies of the consumer durables for the market. It was found that the three major companies A, B, and C have market share of 35%, 25%, and 40% respectively out of which 2%, 1%, and 3% are not up to the satisfaction. A consumer buys one product and is dissatisfied with it. What is the probability that it might be from the company C?

(M.D.U. May 2006)

10. By examining the chest X-ray, the probability that T.B. is detected when a person is actually suffering is 0.9. The probability that the doctor diagnoses incorrectly that a person has T.B. on the basis of X-ray is 0.001. In a certain city, 1 in 1000 persons suffer from T.B. A person is selected at random and is diagnosed to have T.B. What is the chance that he actually has T.B.?

11. Assume that the chance of a patient having a heart attack is 10%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga.

12. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads. What is the probability that it was the two headed coin?

Answers

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5.16. RANDOM VARIABLE

If the numerical values assumed by a variable are the result of some chance factors, so that a particular value cannot be exactly predicted in advance, the variable is then called a random variable. A random variable is also called 'chance variable' or 'stochastic variable'.

Random variables are denoted by capital letters, usually, from the last part of the alphabet, for instance, X, Y, Z etc.

Continuous and Discrete Random Variables

A continuous random variable is one which can assume any value within an interval, i.e., all values of a continuous scale. For example (i) the weights (in kg) of a group of individuals, (ii) the heights of a group of individuals.

A discrete random variable is one which can assume only isolated values. For example, (i) the number of heads in 4 tosses of a coin is a discrete random variable as it cannot assume values other than 0, 1, 2, 3, 4. (ii) the number of aces in a draw of 2 cards from a well shuffled deck is a random variable as it can take the values 0, 1, 2 only.

PROBABILITY DISTRIBUTIONS

5.17. (a) DISCRETE PROBABILITY DISTRIBUTION

Let a random variable X assume values \(x_1, x_2, x_3, ..., x_n\) with probabilities \(p_1, p_2, p_3, ..., p_n\) respectively, where \(P(X = x_i) = p_i \geq 0\) for each \(x_i\) and \(p_1 + p_2 + p_3 + ... + p_n = 1\).

Then

\[
X : x_1, x_2, x_3, ..., x_n
\]

\[
P(X) : p_1, p_2, p_3, ..., p_n
\]

is called the discrete probability distribution for X and it spells out how a total probability of 1 is distributed over several values of the random variable.

5.17. (b) CONTINUOUS PROBABILITY DISTRIBUTION

Let X be a continuous random variable taking values in the interval \((-\infty, \infty)\). Let \(f(x)\) be a function satisfying the following properties:

(i) \(f(x)\) is integrable on \((-\infty, \infty)\)

(ii) \(f(x) \geq 0\) for all \(x\) in \((-\infty, \infty)\)

(iii) \(\int_{-\infty}^{\infty} f(x) \, dx = 1\)

Then \(f(x)\) is called the probability distribution (or density) function (p.d.f.) of the continuous random variable X.

The probability for a continuous random variable X to fall in the interval \([a, b]\) is

\[
P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx
\]

which is nothing but the area between the continuous curve \(y = f(x)\), \(x = a, x = b\) and x-axis. The curve \(y = f(x)\) is called the probability curve.

Taking \(b = a\), \(P(X = a) = \int_{a}^{a} f(x) \, dx = 0\). Also,

\[
P(a \leq X < b) = P(a < X < b) = P(a < X = b) = P(a < X < b)
\]

If X is a continuous random variable with p.d.f. \(f(x)\), then

\[
F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x) \, dx
\]

is called cumulative distribution function or simply distribution function of the continuous random variable X. The distribution function has the following properties:

(i) \(0 \leq F(x) \leq 1, -\infty < x < \infty\)

(ii) \(F(x) = f(x) \geq 0\)

(iii) \(P(a \leq X < b) = \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{a} f(x) \, dx = P(X \leq b) - P(X \leq a) = F(b) - F(a)\)
5.18. MEAN AND VARIANCE OF RANDOM VARIABLES

(a) Let \( X: x_1, x_2, x_3, \ldots, x_n \)

be a discrete probability distribution.

We denote the mean by \( \mu \) and define

\[
\mu = \frac{\sum x_i p_i}{\sum p_i} = \sum x_i p_i
\]

Other names for the mean are average or expected value \( E(X) \).

We denote the variance by \( \sigma^2 \) and define

\[
\sigma^2 = \sum x_i (x_i - \mu)^2 p_i
\]

If \( \mu \) is not a whole number, then

\[
\sigma^2 = \sum x_i^2 p_i - \mu^2
\]

Standard deviation \( \sigma = \sqrt{\text{Variance}} \).

The \( r \)th moment about the mean is denoted by \( \mu_r \) and defined as

\[
\mu_r = \sum (x_i - \mu)^r f(x_i)
\]

Putting, \( r = 0, 1, 2, 3 \) and \( 4 \), we get

\[
\mu_0 = \sum f(x_i) = \sum p_i = 1
\]

\[
\mu_1 = \sum (x_i - \mu) f(x_i) = \sum (x_i - \mu) p_i = \mu - \mu = 0
\]

\[
\mu_2 = \sum (x_i - \mu)^2 f(x_i) = \sigma^2
\]

\[
\mu_3 = \sum (x_i - \mu)^3 f(x_i)
\]

\[
\mu_4 = \sum (x_i - \mu)^4 f(x_i)
\]

In practice the first four moments suffice.

Mean deviation from mean = \( \sum |x_i - \mu| f(x_i) \)

(b) If \( X \) is a continuous random variable with probability density function \( f(x) \), then

\[
E(X) = \int_{-\infty}^{\infty} x f(x) \, dx
\]

\[
E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx
\]

\[
\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx = E(X^2) - [E(X)]^2
\]

Mean deviation from mean = \( \int_{-\infty}^{\infty} |x - \mu| f(x) \, dx \)

The \( r \)th moment about the mean is defined as

\[
\mu_r = \int_{-\infty}^{\infty} (x - \mu)^r f(x) \, dx
\]

PROBABILITY DISTRIBUTIONS

Moments about the mean are called central moments.

(c) Moment Generating Function

Consider the function \( M_X(t) = \sum p_i e^{tx_i} \cdot e^{-t} \)

This function is a function of the parameter \( t \) and gives the mean of the probability distribution of \( e^{x_i} \cdot e^{-t} \). Expanding the exponential in equation (1), we get

\[
M_X(t) = \sum p_i \left[ 1 + tx_i - \frac{t^2}{2!} (x_i - \mu)^2 + \ldots + \frac{t^r}{r!} (x_i - \mu)^r + \ldots \right]
\]

\[
= \sum p_i + t\sum p_i (x_i - \mu) + \frac{t^2}{2!} \sum p_i (x_i - \mu)^2 + \ldots + \frac{t^r}{r!} \sum p_i (x_i - \mu)^r + \ldots
\]

where \( \mu_r \) is the moment of order \( r \) about \( \mu \). Thus \( M_X(t) \) generates moments and is, therefore, called the moment generating function (m.g.f.) of the discrete probability distribution of the variate \( X \) about the value \( x = \mu \). We observe that \( \mu_r \) is equal to the co-efficient of \( \frac{t^r}{r!} \) in the expansion of m.g.f. \( M_X(t) \).

Alternately, \( \mu_r \) can be obtained by differentiating \( M_X(t) \) \( r \) times with respect to \( t \) and then putting \( t = 0 \).

Thus,

\[
\mu_r = \left. \frac{d^r}{dt^r} M_X(t) \right|_{t=0}
\]

Rewriting equation (1) as

\[
M_X(t) = \sum p_i e^{tx_i} = e^{\mu t} \sum p_i e^{t(x_i - \mu)} = e^{\mu t} M_X(t)
\]

Hence the m.g.f. about the value \( \mu \) is \( e^{\mu t} \) times the m.g.f. about the origin.

If \( X \) is a continuous random variable with probability density function \( f(x) \), then

\[
M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx
\]

Note 1. If \( X \) is a random variable and \( a, b \) are constants, then

\[
E(aX + b) = aE(X) + b
\]

\[
\text{Var}(aX + b) = a^2 \text{Var}(X)
\]

2. Relation between central moments and moments about any arbitrary origin \( a \)

\[
\mu'_1 = \mu - a
\]

\[
\mu'_2 = \mu^2 - 2\mu a + a^2
\]

\[
\mu'_3 = 3\mu^3 - 6\mu^2 a + 6\mu a^2 - 3a^3
\]

\[
\mu'_4 = 4\mu^4 - 12\mu^3 a + 12\mu^2 a^2 - 6\mu a^3 + 3a^4
\]
ILLUSTRATIVE EXAMPLES

Example 1. Five defective bulbs are accidentally mixed with twenty good ones. It is not possible to just look at a bulb and tell whether or not it is defective. Find the probability distribution of the number of defective bulbs, if four bulbs are drawn at random from this lot.

Sol. Let $X$ denote the number of defective bulbs in 4. Clearly $X$ can take the values 0, 1, 2, 3 or 4.

- Number of defective bulbs = 5
- Number of good bulbs = 20
- Total number of bulbs = 25

\[
P(X = 0) = \frac{\binom{20}{4}}{\binom{25}{4}} = \frac{20 \times 19 \times 18 \times 17}{25 \times 24 \times 23 \times 22} = 0.969
\]

\[
P(X = 1) = \frac{\binom{5}{1} \times \binom{20}{3}}{\binom{25}{4}} = \frac{5 \times 20 \times 19 \times 18}{25 \times 24 \times 23 \times 22} = 0.1140
\]

\[
P(X = 2) = \frac{\binom{5}{2} \times \binom{20}{2}}{\binom{25}{4}} = \frac{10 \times 20 \times 19}{25 \times 24 \times 23 \times 22} = 0.380
\]

\[
P(X = 3) = \frac{\binom{5}{3} \times \binom{20}{1}}{\binom{25}{4}} = \frac{10 \times 20}{25 \times 24 \times 23 \times 22} = 0.040
\]

\[
P(X = 4) = \frac{\binom{5}{4} \times \binom{20}{0}}{\binom{25}{4}} = \frac{5}{25 \times 24 \times 23 \times 22} = 0.001
\]

- The probability distribution of the random variable $X$ is:

<table>
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<th>$p$</th>
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<tbody>
<tr>
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</tr>
<tr>
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</tr>
<tr>
<td>2</td>
<td>0.380</td>
</tr>
<tr>
<td>3</td>
<td>0.040</td>
</tr>
<tr>
<td>4</td>
<td>0.001</td>
</tr>
</tbody>
</table>

To find the mean and variance:

\[
\mu = \sum x \cdot p(x) = 1
\]

\[
\sigma^2 = \sum (x - \mu)^2 \cdot p(x) = \frac{5}{3} - \frac{1}{3} \cdot \frac{2}{3}
\]

Example 2. A die is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and the variance of the number of successes.

Sol. Let $X$ denote the number of success. Clearly $X$ can take the values 0, 1, 2 or 3.

- Probability of success = \(\frac{2}{6} = \frac{1}{3}\)
- Probability of failure = \(1 - \frac{1}{3} = \frac{2}{3}\)

\[
P(X = 0) = \left(\frac{2}{3}\right)^3 = \frac{8}{27}
\]

\[
P(X = 1) = 3 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{12}{27}
\]

\[
P(X = 2) = 3 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{6}{27}
\]

\[
P(X = 3) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}
\]

Example 3. A random variable $X$ has the following probability function:

Values of $X$ | $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7
--- | --- | --- | --- | --- | --- | --- | --- | --- | ---
$p(x)$ | $p$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7
(i) Find $k$.
(ii) Evaluate $P(X < 6)$, $P(X \geq 6)$, $P(3 \leq X \leq 6)$
(iii) Find the minimum value of $x$ so that $P(X \geq x) > \frac{1}{2}$.

Sol. (i) Since $\sum_{x=0}^{7} p(x) = 1$, we have

\[
0 + k + 2k + 3k + 2k^2 + 7k^3 + k = 1
\]

\[
= 10k^2 + 9k - 1 = 0 \quad \Rightarrow \quad (10k - 1)(k + 1) = 0
\]

$\Rightarrow \quad k = \frac{1}{10}$

(ii) $P(X < 6) = P(X = 0) + P(X = 1) + \ldots + P(X = 5)$

\[
= 0 + 1 + 2 + 3 + 4 + 5 = \frac{8}{10} + \frac{1}{10} = \frac{81}{100}
\]

$P(X \geq 6) = P(X = 6) + P(X = 7)$

\[
= 2k^2 + 7k^3 + k = \frac{9}{100} + \frac{1}{10} = \frac{1}{100}
\]

$P(3 \leq X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6)$

\[
= \frac{3}{10} + \frac{3}{100} = \frac{33}{100}
\]
(iii) \( P(X \leq 1) = k = \frac{1}{10} < \frac{1}{2} \)

\[
P(X \leq 2) = k + 2k = \frac{3}{10} < \frac{1}{2}
\]

\[
P(X \leq 3) = k + 2k + 2k = \frac{8}{10} < \frac{1}{2}
\]

\[\therefore \text{ The maximum value of } x \text{ so that } P(X \leq x) > \frac{1}{2} \text{ is } 4.\]

**Example 4.** In a lottery, \( m \) tickets are drawn at a time out of \( n \) tickets numbered from 1 to \( n \). Find the expected value of the sum of the numbers on the tickets drawn.

(M.D.U. May 2011)

Sol. Let \( X \) denote the number on a ticket then \( X \) assumes values 1, 2, 3, ..., \( n \).

The probability of drawing a ticket out of \( n \) tickets is \( \frac{1}{n} \).

\[\therefore \text{ Values of } X, \quad x_i : 1 \quad 2 \quad 3 \quad \ldots \quad n
\]

\[P(X = x) = \frac{1}{n} \quad \frac{1}{n} \quad \frac{1}{n} \quad \ldots \quad \frac{1}{n}
\]

\[
E(x) = \sum p_i x_i
\]

\[
= 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} + \ldots + n \cdot \frac{1}{n}
\]

\[
= \frac{1}{n} \left( 1 + 2 + 3 + \ldots + n \right)
\]

\[
= \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}
\]

If the numbers on the \( m \) tickets drawn are \( n_1, n_2, \ldots, n_m \) where \( n_i \in \{1, 2, 3, \ldots, n\} \), \( i = 1, 2, \ldots, m \), then the expected value of the sum of numbers on \( m \) tickets

\[E(n_1) + E(n_2) + \ldots + E(n_m)
\]

\[m \cdot E(X) = \frac{m(n+1)}{2}
\]

**Example 5.** Is the function defined as follows a density function?

\[f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}
\]

If so, find \( P(1 \leq X \leq 2) \).

Sol. \( f(x) \geq 0 \) for every \( x \) in \( (-\infty, \infty) \) and

\[\int_{-\infty}^{0} f(x) \, dx + \int_{0}^{\infty} f(x) \, dx
\]

\[= \int_{0}^{\infty} 0 \, dx + \int_{0}^{\infty} e^{-x} \, dx = 0 + \left[ - \frac{e^{-x}}{-1} \right]_{0}^{\infty} = - (0 - 1) = 1
\]

Since \( f(x) \) satisfies the requirements for a density function, therefore, \( f(x) \) is a density function.

\[
P(1 \leq X \leq 2) = \int_{1}^{2} e^{-x} \, dx = \left[ -e^{-x} \right]_{1}^{2} = e^{-1} - e^{-2}
\]

\[= 0.368 - 0.135 = 0.233.
\]

**Example 6.** \( X \) is a continuous random variable with probability density function given by

\[
f(x) = \begin{cases} kx, & 0 \leq x < 2 \\ 2k, & 2 \leq x < 4 \\ -kx + 6k, & 4 \leq x < 8 \end{cases}
\]

Find \( k \) and mean value of \( X \).

Sol. Since total probability = 1, we have

\[
\int_{0}^{8} f(x) \, dx = 1
\]

\[
\int_{0}^{2} f(x) \, dx + \int_{2}^{4} f(x) \, dx + \int_{4}^{8} f(x) \, dx = 1
\]

\[
\int_{0}^{2} kx \, dx + \int_{2}^{4} 2k \, dx + \int_{4}^{8} (-kx + 6k) \, dx = 1
\]

\[
\int_{0}^{2} kx \, dx + \left[ 2kx \right]_{2}^{4} + \left[ -k \frac{x^2}{2} + 6k \right]_{4}^{8}
\]

\[
= k \left( \frac{x^2}{2} \right)_{0}^{2} + k \left( \frac{x^2}{2} \right)_{2}^{4} + k \left( \frac{x^2}{2} \right)_{4}^{8}
\]

\[
= k (2 - 0) + 2k (4 - 2) + (-18k + 36k) = -8k + 24k = 16k = 1
\]

\[8k = 1 \therefore k = \frac{1}{8}
\]

Mean value of \( X = E(X) = \int_{0}^{8} x \, f(x) \, dx
\]

\[
= \int_{0}^{2} x \cdot kx \, dx + \int_{2}^{4} x \cdot 2k \, dx + \int_{4}^{8} x (-kx + 6k) \, dx
\]

\[
= k \left[ \frac{x^3}{3} \right]_{0}^{2} + 2k \left[ \frac{x^2}{2} \right]_{2}^{4} + \left[ -k \frac{x^3}{3} + 6k \cdot \frac{x^2}{2} \right]_{4}^{8}
\]

\[
= k \left( \frac{8}{3} \right) + k (12) - k (152) + 3k (20)
\]

\[= 24k = 24 \left( \frac{1}{8} \right) = 3.
\]
Example 7. The density function of a random variable \( X \) is given by \( f(x) = k(x - 2), 0 \leq x \leq 2 \). Find (i) \( k \) (ii) mean (iii) variance (iv) mean deviation about the mean.

**Sol.** (i) Since total probability = 1, we have

\[
\int_0^2 f(x) \, dx = 1
\]

\[
\Rightarrow \int_0^2 k(x - 2) \, dx = 1 \Rightarrow k \left[ \frac{x^2 - x^3}{2} \right]_0^2 = 1
\]

\[
\Rightarrow k \left( 4 - \frac{8}{3} \right) = 1 \Rightarrow k \cdot \frac{4}{3} = 1 \Rightarrow k = \frac{3}{4}
\]

Hence

\( f(x) = \frac{3}{4} x(2 - x), 0 \leq x \leq 2 \).

(ii) Mean = \( E(X) = \int_0^2 x f(x) \, dx = \int_0^2 \frac{3}{4} x^2 \, dx \)

\[
= \frac{3}{4} \left[ \frac{x^3}{3} \right]_0^2 = \frac{3}{4} \left[ \frac{8}{3} - 4 \right] = \frac{3}{4} \cdot \frac{4}{3} = 1
\]

(iii) Var \( (X) = E(X^2) - [E(X)]^2 \)

\[
\int_0^2 x^2 f(x) \, dx - (1)^2 = \frac{3}{4} \int_0^2 x^2 (2 - x) \, dx - 1
\]

\[
= \frac{3}{4} \left[ \frac{2x^3 - x^4}{4} \right]_0^2 = \frac{3}{4} \left[ \frac{16}{4} - 4 \right] = \frac{3}{4} \cdot \frac{4}{3} = 1
\]

\[
\frac{3}{4} \left[ \frac{8}{5} - 1 \right] = \frac{3}{5} = 1
\]

(iv) Mean deviation about the mean

\[
\int_0^2 |x - \bar{x}| f(x) \, dx = \int_0^2 |x - 1| f(x) \, dx
\]

\[
= \int_0^2 |x - 1| f(x) \, dx + \int_1^2 |x - 1| f(x) \, dx
\]

\[
= \int_0^2 (x - 1) \frac{3}{4} x(2 - x) \, dx + \int_1^2 (x - 1) \frac{3}{4} x(2 - x) \, dx
\]

\[
= \frac{3}{4} \int_0^2 (2x - 3x^2 + x^3) \, dx + \frac{3}{4} \int_0^1 (2x - 3x^2 + x^3) \, dx
\]

\[
= \frac{3}{4} \int_0^2 x^2 + x^3 \, dx + \frac{3}{4} \int_0^1 x^2 + x^3 \, dx
\]

\[
= \frac{3}{4} \frac{1}{4} + \frac{3}{4} \frac{1}{4} = \frac{3}{8}
\]

EXERCISE 5.3

1. Two bad eggs are mixed accidently with 10 good ones. Find the probability distribution of the number of bad eggs in 3, drawn at random, without replacement, from this lot.

2. A die is tossed twice. Getting a number greater than 4 is considered a success. Find the variance of the probability distribution of the number of successes.

3. Two cards are drawn simultaneously from a well-shuffled deck of 52 cards. Compute the variance for the number of aces.

4. An urn contains 4 white and 3 red balls. Three balls are drawn, with replacement, from this urn. Find \( p \), \( \sigma^2 \) and \( \sigma \) for the number of red balls drawn.

5. Compute the variance of the probability distribution of the number of doubles in four throws of a pair of dice. (M.D.U. May 2006)

6. Four coins are tossed. What is the expectation of the number of heads?

7. Suppose that \( X \) is a random variable for which \( E(X) = 10 \) and \( Var(X) = 25 \). Find the positive values of \( a \) and \( b \) such that \( Y = aX - b \) has expectation 0 and variance 1.

8. A variate \( X \) has the probability distribution

\[
\begin{array}{cc}
X & P(X) \\
-3 & 1/6 \\
-1 & 1/2 \\
1 & 1/3 \\
\end{array}
\]

Find \( E(X) \) and \( E(X^2) \). Hence evaluate \( E(2X + 1)^2 \).
A random variable $X$ has the following probability distribution:

$$
\begin{align*}
x & \quad 0 & 1 & 2 & 3 \\
p(x) & \quad 0.1 & 0.2 & 0.3 & 0.4
\end{align*}
$$

Find the value of $k$ and calculate mean and variance.

9. From an urn containing 3 red and 2 white balls, a man is to draw 2 balls at random without replacement, being promised ₹ 20 for each red ball he draws and ₹ 10 for each white one. Find his expectation.

10. A random variable $X$ has the following probability distribution:

$$
\begin{align*}
x & \quad 5 & 6 & 7 & 8 \\
p(x) & \quad 0.3 & 0.1 & 0.2 & 0.4
\end{align*}
$$

(i) Determine the value of $a$.

(ii) Find $P(X < 3)$, $P(X \geq 3)$, $P(2 \leq X < 5)$.

(iii) What is the smallest value of $x$ for which $P(X = x) > 0.5$?

11. Find the standard deviation for the following discrete distribution:

$$
\begin{align*}
x & \quad 8 & 12 & 16 & 20 & 24 \\
p(x) & \quad 0.1 & 0.2 & 0.3 & 0.4 & 0.2
\end{align*}
$$

13. Is the function given below a density function?

$$
f(x) = \begin{cases} 
0 & \text{for } x < 2 \\
\frac{1}{18} (3 + 2x) & \text{for } 2 \leq x \leq 4 \\
0 & \text{for } x > 4 
\end{cases}
$$

Also find $P(2 \leq X \leq 3)$.

14. Find the mean and variance of the following density function:

$$
f(x) = \begin{cases} 
x & \text{for } 0 < x < 1 \\
2 - x & \text{for } 1 < x < 2 \\
0 & \text{otherwise}
\end{cases}
$$

15. The probability density $p(x)$ of a continuous random variable is given by

$$p(x) = \frac{1}{2} e^{-x} - x < x < \infty.$$

Prove that $\gamma_0 = \frac{1}{2}$. Find the mean and variance of the distribution.

5.19. THEORETICAL DISTRIBUTIONS

Frequency distributions can be classified under two heads:

(i) Observed Frequency Distributions.

(ii) Theoretical or Expected Frequency Distributions.

Observed frequency distributions are based on actual observation and experimentation. If a certain hypothesis is assumed, it is sometimes possible to derive mathematically what the frequency distribution of certain universe should be. Such distributions are called Theoretical Distributions.

There are many types of theoretical frequency distributions but we shall consider only three which are of great importance:

(i) Binomial Distribution (or Bernoulli's Distribution):

(ii) Poisson's Distribution;

(iii) Normal Distribution.

BINOMIAL (OR BERNOULLI'S) DISTRIBUTION

5.20. BINOMIAL PROBABILITY DISTRIBUTION

Let there be $n$ independent trials in an experiment. Let a random variable $X$ denote the number of successes in these $n$ trials. Let $p$ be the probability of a success and $q$ that of a failure in a single trial so that $p + q = 1$. Let the trials be independent and $p$ be constant for every trial.

Let us find the probability of $r$ successes in $n$ trials.

$r$ successes can be obtained in $n$ trials in $\binom{n}{r}$ ways.

\[
P(X = r) = \binom{n}{r} p^r q^{n-r} 
\]

\[
= \binom{n}{r} p^r q^{n-r}
\]

Answers

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(X)$</td>
<td>0.12</td>
<td>0.29</td>
<td>0.22</td>
</tr>
</tbody>
</table>

\[
= \binom{n}{r} p^r q^{n-r}
\]
Hence \( P(X = r) = \binom{n}{r} q^{n-r} p^r \), where \( p + q = 1 \) and \( r = 0, 1, 2, \ldots, n \).

The distribution (1) is called the \textit{binomial probability distribution} and \( X \) is called the \textit{binomial variate}.

\textbf{Note 1.} \( P(X = r) \) is usually written as \( P(r) \).

\textbf{Note 2.} The successive probabilities \( P(r) \) in (1) for \( r = 0, 1, 2, \ldots, n \) are

\[ \binom{n}{r} q^{n-r} p^r, \binom{n}{r+1} q^{n-r-1} p^{r+1}, \ldots, \binom{n}{n} q^0 p^n, \]

which are the successive terms of the binomial expansion of \((q + p)^n\). That is why this distribution is called "binomial" distribution.

\textbf{Note 3.} \( n \) and \( p \) occurring in the binomial distribution are the \textbf{parameters} of the distribution.

\textbf{Note 4.} In a binomial distribution,

(i) the number of trials is finite.

(ii) each trial has only two possible outcomes usually called success and failure.

(iii) all the trials are independent.

(iv) \( p \) (and hence \( q \)) is constant for all the trials.

\textbf{Note 5.} A binomial distribution with \( n \) trials and probability of success in each trial as \( p \), is denoted by \( B(n, p) \).

\section*{5.21. Recurrence or Recursion Formula for the Binomial Distribution}

In a binomial distribution,

\[ P(r) = \binom{n}{r} q^{n-r} p^r, \quad P(r+1) = \binom{n}{r+1} q^{n-r-1} p^{r+1} = \frac{n!}{(n-r-1)!r!} q^{n-r-1} p^{r+1}. \]

Thus,

\[ \frac{P(r+1)}{P(r)} = \frac{(n-r)!}{(n-r-1)!} \cdot \frac{(r+1)!}{r!} \cdot \frac{q}{p} = \frac{(n-r)(n-r-1)!}{(n-r-1)!} \cdot \frac{r+1}{r} \cdot \frac{p}{q} = \frac{n-r}{r+1} \cdot \frac{p}{q} \cdot \frac{P(r)}{P(r+1)}. \]

\[ P(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q} \cdot P(r), \]

which is the required recurrence formula. Applying this formula successively, we can find \( P(0), P(1), P(2), \ldots, \) if \( P(0) \) is known.

\section*{5.22. Mean and Variance of the Binomial Distribution \textit{(U.P.T.U. 2008)}}

For the binomial distribution, \( P(r) = \binom{n}{r} q^{n-r} p^r \),

\[ \text{Mean } \mu = \sum_{r=0}^{n} r \binom{n}{r} q^{n-r} p^r = 0 \cdot \binom{n}{0} q^n p^0 + 1 \cdot \binom{n}{1} q^{n-1} p^1 + 2 \cdot \binom{n}{2} q^{n-2} p^2 + \ldots + n \cdot \binom{n}{n} q^0 p^n = np \cdot q^{n-1} p + \frac{n(n-1)}{2.1} q^{n-2} p^2 + \frac{n(n-1)(n-2)}{3.2.1} q^{n-3} p^3 + \ldots + np^n. \]

Hence the mean of the binomial distribution is \( np \).

\[ \text{Variance } \sigma^2 = \sum_{r=0}^{n} r^2 P(r) - \mu^2 = \sum_{r=0}^{n} [r(r+1)] P(r) - \mu^2 = npq + \frac{n(n-1)}{2.1} q^{n-2} p^2 + \frac{n(n-1)(n-2)}{3.2.1} q^{n-3} p^3 + \ldots + np^n. \]

\[ \text{Hence the variance of the binomial distribution is } npq. \]

\[ \text{Standard deviation of the binomial distribution is } \sqrt{npq}. \]

Similarly, we can prove that

\[ \beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{(1-2p)^2 p^4 + 3 \cdot (1-2p)^2 p^3 + \ldots + n \cdot (1-2p)^2 p^n}{npq^2} = \frac{(1-2p)^2 p^4}{npq^2}. \]

Hence

\[ \gamma_1 = \frac{\beta_2}{\mu_2} = \frac{1-2p}{\sqrt{npq}}. \]

\[ \text{Note.} \quad \gamma_1 = \frac{\beta_2}{\mu_2} = \frac{1-2p}{\sqrt{npq}} \text{ gives a measure of skewness of the binomial distribution. If } p < \frac{1}{2}, \text{ skewness is positive, if } p > \frac{1}{2}, \text{ skewness is negative and if } p = \frac{1}{2}, \text{ it is zero.} \]

\[ \beta_2 = 3 + \frac{1-2p}{\sqrt{npq}} \text{ gives a measure of the kurtosis of the binomial distribution.} \]

\section*{ILLUSTRATIVE EXAMPLES}

**Example 1.** During war, 1 ship out of 9 was sunk on an average in making a certain voyage. What was the probability that exactly 3 out of a convoy of 6 ships would arrive safely?

\[ \text{Sol.} \quad p = \frac{8}{9}, \quad q = \frac{1}{9}, \quad n = 6. \]

\[ \text{Binomial distribution is } \left( \begin{array}{c} 1 \\ \frac{8}{9} \end{array} \right) \]

\[ \text{The probability that exactly 3 ships arrive safely } = \binom{6}{3} \left( \frac{1}{9} \right)^3 \left( \frac{8}{9} \right)^3 = \frac{10240}{9^6}. \]
Example 2. Assume that on the average one telephone number out of fifteen called between 2 P.M. and 3 P.M. on week-days is busy. What is the probability that if 6 randomly selected telephone numbers are called (i) not more than three, (ii) at least three of them will be busy?

Sol. $p$, the probability of a telephone number being busy between 2 P.M. and 3 P.M. on week-days $= \frac{1}{15}$

$q = 1 - \frac{1}{15} = \frac{14}{15}$ $n = 6$; Binomial distribution is $\binom{6}{k} \left(\frac{1}{15}\right)^k \left(\frac{14}{15}\right)^{6-k}$

The probability that not more than three will be busy

$= P(0) + P(1) + P(2) + P(3)$

$= \binom{6}{0} \left(\frac{14}{15}\right)^6 + \binom{6}{1} \left(\frac{1}{15}\right) \left(\frac{14}{15}\right)^5 + \binom{6}{2} \left(\frac{1}{15}\right)^2 \left(\frac{14}{15}\right)^4 + \binom{6}{3} \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3$

$= \left(\frac{14}{15}\right)^6 + 6 \left(\frac{1}{15}\right) \left(\frac{14}{15}\right)^5 + 15 \left(\frac{1}{15}\right)^2 \left(\frac{14}{15}\right)^4 + 20 \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3$

$= \left(\frac{14}{15}\right)^6 \times 1176 + 210 = 2744 \times 4150 = 0.9997$

The probability that at least three of them will be busy

$= P(3) + P(4) + P(5) + P(6)$

$= \binom{6}{3} \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3 + \binom{6}{4} \left(\frac{1}{15}\right)^4 \left(\frac{14}{15}\right)^2 + \binom{6}{5} \left(\frac{1}{15}\right)^5 \left(\frac{14}{15}\right) + \binom{6}{6} \left(\frac{1}{15}\right)^6 = 0.005$

Example 3. Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six?

Sol. $p =$ the chance of getting 5 or 6 with one die $= \frac{2}{6} = \frac{1}{3}$

$q = 1 - \frac{1}{3} = \frac{2}{3}$ $n = 6, N = 729$

since dice are in sets of 6 and there are 729 sets.

The binomial distribution is $N(q + p)^n = 729 \left(\frac{1}{3} \frac{1}{3}\right)^6$

The expected number of times at least three dice showing five or six

$= 729 \left[ \binom{6}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + \binom{6}{4} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + \binom{6}{5} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + \binom{6}{6} \left(\frac{1}{3}\right)^6 \right]$

$= 729 \left[ \left(\frac{1}{3}\right)^3 + 7 \left(\frac{1}{3}\right)^4 + 7 \left(\frac{1}{3}\right)^5 + 1 \left(\frac{1}{3}\right)^6 \right]$

$= 729 \left[ \frac{1}{27} + \frac{7}{81} + \frac{7}{81} + \frac{1}{81} \right] = 233.$

Example 4. Out of 800 families with 2 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) at least one boy (iii) no girl (iv) at most two girls?

Assume equal probabilities for boys and girls.

(M.D.U. Dec. 2010)

Example 5. Suppose $X$ has a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome.

Sol. Here $n = 6, \mu = \frac{1}{2} \Rightarrow q = 1 - \mu = \frac{1}{2}$

$P(0) = \binom{6}{0} \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^6 = \frac{1}{64}$

$P(1) = \binom{6}{1} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^1 = \frac{6}{64}$

$P(2) = \binom{6}{2} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = \frac{15}{64}$

$P(3) = \binom{6}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3 = \frac{20}{64}$

$P(4) = \binom{6}{4} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^4 = \frac{15}{64}$

$P(5) = \binom{6}{5} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^5 = \frac{6}{64}$

$P(6) = \binom{6}{6} \left(\frac{1}{2}\right)^6 = \frac{1}{64}$

Since $P(3)$ is the maximum among all $P(r), r = 0, 1, 2, 3, 4, 5, 6$, therefore, $X = 3$ is the most likely outcome.

Example 6. Find the mean of the binomial distribution $B\left(4, \frac{1}{3}\right)$.

Sol. Let $X$ be the random variable whose probability distribution is $B\left(4, \frac{1}{3}\right)$ Then

$\mu = 0, 1, 2, 3, 4.$
Here \( n = 4 \), \( p = \frac{1}{3} \), \( q = 1 - \frac{1}{3} = \frac{2}{3} \).

\[
P(0) = \binom{4}{0} \left( \frac{1}{3} \right)^0 \left( \frac{2}{3} \right)^4 = \frac{16}{81};
\]

\[
P(1) = \binom{4}{1} \left( \frac{1}{3} \right)^1 \left( \frac{2}{3} \right)^3 = \frac{32}{81};
\]

\[
P(2) = \binom{4}{2} \left( \frac{1}{3} \right)^2 \left( \frac{2}{3} \right)^2 = \frac{24}{81};
\]

\[
P(3) = \binom{4}{3} \left( \frac{1}{3} \right)^3 \left( \frac{2}{3} \right)^1 = \frac{8}{81};
\]

\[
P(4) = \binom{4}{4} \left( \frac{1}{3} \right)^4 \left( \frac{2}{3} \right)^0 = \frac{1}{81}.
\]

Mean (\( \mu \)) = \( \sum_{x=0}^{n} x \cdot p(x) = \frac{16}{81} \times 0 + \frac{32}{81} \times 1 + \frac{24}{81} \times 2 + \frac{8}{81} \times 3 + \frac{1}{81} \times 4 = \frac{108}{81} = \frac{4}{3} \).

Example 7. The probability of a shooter hitting a target is \( \frac{3}{4} \). How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 0.99? 

Sol. Let the shooter fire \( n \) times. Here \( n \) fires are \( n \) Bernoulli trials with

\[
p = \frac{3}{4} \quad \text{and} \quad q = 1 - \frac{3}{4} = \frac{1}{4}.
\]

Now \( P(X \geq 1) > 0.99 \)

\[
\Rightarrow 1 - P(X = 0) > 0.99
\]

\[
\Rightarrow P(X = 0) < 0.01
\]

\[
\Rightarrow \binom{n}{0} \left( \frac{1}{4} \right)^0 \left( \frac{3}{4} \right)^n < 0.01
\]

\[
\Rightarrow 4^n > 100.
\]

The minimum value of \( n \) satisfying (1) is 4.

Hence the shooter must fire 4 times.

Example 8. Fit a binomial distribution to the following data:

\[
x: 0 \quad 1 \quad 2 \quad 3 \quad 4
\]

\[
f: 30 \quad 62 \quad 46 \quad 10 \quad 2
\]

Sol. The table is as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f )</th>
<th>( fx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>62</td>
<td>62</td>
</tr>
<tr>
<td>2</td>
<td>46</td>
<td>92</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

\( \Sigma f = 150 \)

\( \Sigma fx = 192 \)

EXERCISE 5.4

1. Ten coins are tossed simultaneously. Find the probability of getting at least seven heads.

2. The probability of any ship of a company being destroyed on a certain voyage is 0.02. The company owns 6 ships for the voyage. What is the probability of:
   (i) losing one ship
   (ii) losing at most two ships
   (iii) losing none.

3. (a) The probability that a man aged 60 will live to be 70 is 0.65. What is the probability that out of ten men now aged 60, at least 7 would live to be 70?

(b) The probability that a man aged 50 years will die within a year is 0.01125. What is the probability that out of 12 such men, at least 11 will reach their fifty first birthday?

(M.D.U. Dec. 2011)

4. (a) The incidence of occupational disease in an industry is such that the workers have a 23% chance of suffering from it. What is the probability that out of six workers chosen at random, four or more will suffer from the disease?

(b) The incidence of occupational disease in an industry is such that the workers have a 10% chance of suffering from it. What is the probability that in a group of 7, five or more will suffer from it?


5. The probability that a pen manufactured by a company will be defective is \( \frac{1}{10} \). If 12 such pens are manufactured, find the probability that
   (i) exactly two will be defective
   (ii) at least two will be defective
   (iii) none will be defective.

6. If the chance that one of the ten telephone lines is busy at an instant is 0.2.
   (i) What is the chance that 5 of the lines are busy?
   (ii) What is the probability that all the lines are busy?

If on an average, 1 vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive, at least 4 will arrive safely.

(P.T.U. 2005)

8. A product is 0.5% defective and is packed in cartons of 100. What percentage contains not more than 3 defectives?

9. A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn, one by one, with replacement, what is the probability that
   (i) none is white
   (ii) all are white
   (iii) at least one is white
   (iv) only two are white?

10. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is \( \frac{5}{6} \). What is the probability that he will knock down fewer than 2 hurdles?
5.23. POISSON DISTRIBUTION AS A LIMITING CASE OF BINOMIAL DISTRIBUTION

If the parameters $n$ and $p$ of a binomial distribution are known, we can find the distribution. But in situations where $n$ is very large and $p$ is very small, application of binomial distribution is very labourious. However, if we assume that as $n \to \infty$ and $p \to 0$ such that $np$ always remains finite, say $\lambda$, we get the Poisson approximation to the binomial distribution.

Now, for a binomial distribution

$$P(\text{X} = r) = \binom{n}{r} \, p^r \, (1-p)^{n-r}$$

$$= \frac{n(n-1)(n-2) \ldots (n-r+1)}{r!} \times (1-p)^n \times p^r$$

$$= \frac{n(n-1)(n-2) \ldots (n-r+1)}{r!} \times \left(1 - \frac{\lambda}{n}\right)^n \times \left(\frac{\lambda}{n}\right)^r$$

since $np = \lambda \Rightarrow p = \frac{\lambda}{n}$

$$= \frac{n^r \times (n-1)(n-2) \ldots (n-r+1)}{r!} \times \left(1 - \frac{\lambda}{n}\right)^n \times \left(\frac{\lambda}{n}\right)^r$$

$$= \lambda^r \times \left(\frac{n}{n}\right) \times \left(\frac{n-1}{n}\right) \times \ldots \times \left(\frac{n-r+1}{n}\right) \times \left(1 - \frac{\lambda}{n}\right)^n \times \left(\frac{\lambda}{n}\right)^r$$

As $n \to \infty$, each of the $(r+1)$ factors

$$\left(1 - \frac{\lambda}{n}\right)^n \times \frac{\lambda}{n}$$

tends to 1. Also $\frac{\lambda}{n}$ tends to 1.
Since \( L_n = \frac{1 + \frac{1}{x^n}}{x} e \), the Naperian base. \[ \left( 1 - \frac{\lambda}{n} \right)^n \sim e^{-\lambda} \text{ as } n \to \infty \]

Hence in the limiting case when \( n \to \infty \), we have

\[ P(X = r) = \frac{\lambda^r e^{-\lambda}}{r!} (r = 0, 1, 2, 3, \ldots) \]

where \( \lambda \) is a finite number \( = np \).

(A) represents a probability distribution which is called the Poisson probability distribution.

Note 1. \( \lambda \) is called the parameter of the distribution.

Note 2. \( e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} + \ldots \to \infty \)

Note 3. The sum of the probabilities \( P(r) \) for \( r = 0, 1, 2, 3, \ldots \) is 1, since

\[ P(0) + P(1) + P(2) + P(3) + \ldots = e^{\lambda} + \frac{\lambda e^{\lambda}}{1!} + \frac{\lambda^2 e^{\lambda}}{2!} + \frac{\lambda^3 e^{\lambda}}{3!} + \ldots = e^{\lambda} \left( 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots \right) = e^{\lambda} + \lambda = 1 \]

5.24. Recurrence Formula for the Poisson Distribution

(U.P.T.U. 2006)

For Poisson distribution, \( P(r) = \frac{\lambda^r e^{-\lambda}}{r!} \) and \( P(r+1) = \frac{\lambda^{r+1} e^{-\lambda}}{(r+1)!} \)

\[ \frac{P(r+1)}{P(r)} = \frac{\lambda}{r+1} \text{ or } P(r+1) = \frac{\lambda}{r+1} P(r) \quad r = 0, 1, 2, 3, \ldots \]

This is called the recurrence formula for the Poisson distribution.

5.25. Mean and Variance of the Poisson Distribution

(U.P.T.U. 2006)

For the Poisson distribution, \( P(r) = \frac{\lambda^r e^{-\lambda}}{r!} \)

Mean \( \mu = \sum_{r=0}^{\infty} r P(r) = \sum_{r=0}^{\infty} \frac{\lambda^r e^{-\lambda}}{r!} = \lambda e^{-\lambda} \left( 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots \right) \)

\[ = e^{\lambda} \left( 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots \right) = e^{\lambda} \left( 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots \right) = \lambda = \gamma \]

Thus, the mean of the Poisson distribution is equal to the parameter \( \lambda \).

Variance \( \sigma^2 = \sum_{r=0}^{\infty} r^2 P(r) - \mu^2 = \sum_{r=0}^{\infty} \frac{\lambda^r e^{-\lambda}}{r!} r^2 - \lambda^2 = e^{\lambda} \sum_{r=1}^{\infty} \frac{r^2 \lambda^r}{r!} - \lambda^2 \)

Hence, the variance of the Poisson distribution is also \( \lambda \).

Thus, the mean and the variance of the Poisson distribution are each equal to the parameter \( \lambda \).

Note. The mean and the variance of the Poisson distribution can also be derived from those of the binomial distribution in the limiting case when \( n \to \infty \), \( p \to 0 \) and \( np = \lambda \).

Mean of binomial distribution is \( np \).

\[ \Rightarrow \text{ Mean of Poisson distribution} = \lambda = np = \frac{\lambda}{n} = \frac{\lambda}{n} \]

Variance of binomial distribution is \( np(1-p) \).

\[ \Rightarrow \text{ Variance of Poisson distribution} = \lambda = np(1-p) = \lambda = \left( 1 - \frac{\lambda}{n} \right) = \lambda \]

Illustrative Examples

Example 1. If the variance of the Poisson distribution is 2, find the probabilities for \( r = 1, 2, 3, 4 \) from the recurrence relation of the Poisson distribution.

Sol. \( \lambda \), the parameter of Poisson distribution = Variance = 2

Recurrence relation for the Poisson distribution is:

\[ P(r+1) = \frac{\lambda}{r+1} P(r) \]

Now \( P(r) = \frac{\lambda^r e^{-\lambda}}{r!} \Rightarrow P(0) = e^{-2} = e^{-2} = 0.1353 \)

Putting \( r = 0, 1, 2, 3 \) in (1), we get

\[ \begin{align*}
P(1) &= 2P(0) = 2 \times 0.1353 = 0.2706; \\
P(2) &= \frac{2}{2} P(1) = 0.2706 \\
P(3) &= \frac{2}{3} P(2) = \frac{2}{3} \times 0.2706 = 0.1804; \\
P(4) &= \frac{2}{4} P(3) = \frac{1}{2} \times 0.1804 = 0.0902 \\
\end{align*} \]
Example 2. Assume that the probability of an individual coalminer being killed in a mine accident during a year is $\frac{1}{2400}$. Use Poisson's distribution to calculate the probability that in a mine employing 200 miners there will be at least one fatal accident in a year.

Sol. Here $p = \frac{1}{2400}$, $n = 200$; $\therefore \lambda = np = \frac{200}{2400} = \frac{1}{12} = 0.083$

\[ P(r) = \frac{\lambda^r e^{-\lambda}}{r!} = \frac{(0.083)^r e^{-0.083}}{r!} \]

P(at least one fatal accident) = 1 - P(no fatal accident)

\[ = 1 - P(0) = 1 - \frac{(0.083)^0 e^{-0.083}}{0!} = 1 - 0.92 = 0.08. \]

Example 3. Data was collected over a period of 10 years, showing number of deaths from horse kicks in each of the 290 army corps. The distribution of deaths was as follows:

<table>
<thead>
<tr>
<th>No. of deaths:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency:</td>
<td>109</td>
<td>65</td>
<td>22</td>
<td>3</td>
<td>1</td>
<td>200</td>
</tr>
</tbody>
</table>

Fit a Poisson distribution to the data and calculate the theoretical frequencies.

Sol. Mean of given distribution = $\frac{\sum x}{\sum f} = \frac{65 + 44 + 9 + 4}{200} = \frac{122}{200} = 0.61$.

This is the parameter ($\lambda$) of the Poisson distribution.

Required Poisson distribution is $N = \frac{m^x e^{-m}}{x!}$, where $N = \sum f = 200$

\[ = 200 e^{-0.61} \frac{(0.61)^x}{x!} = 200 \times 0.5435 \frac{(0.61)^x}{x!} = 108.7 \times \frac{(0.61)^x}{x!}. \]

<table>
<thead>
<tr>
<th>$r$</th>
<th>$P(r)$</th>
<th>Theoretical frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>108.7</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>108.7 × 0.61 = 66.3</td>
<td>66</td>
</tr>
<tr>
<td>2</td>
<td>$108.7 \times \frac{(0.61)^2}{2!} = 20.2$</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>$108.7 \times \frac{(0.61)^3}{3!} = 4.1$</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>$108.7 \times \frac{(0.61)^4}{4!} = 0.7$</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>200</td>
<td></td>
</tr>
</tbody>
</table>

Example 4. A car-hire firm has two cars, which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days on which neither car is used and the proportion of days on which some demand is refused. ($e^{-1} = 0.2231$)

(M.D.U. Dec. 2010)

Sol. Since the number of demands for a car is distributed as a Poisson distribution with mean $\lambda = 1.5$.

- Proportion of days on which neither car is used
  $P(0)$ = Probability of there being no demand for the car
  $m^0 e^{-m} = e^{-1.5} = 0.2231$

- Proportion of days on which some demand is refused
  $P(\geq 1)$ = Probability of getting at least one demand
  $\sum_{r=1}^{\infty} P(r) = 1 - P(0) = 1 - e^{-1.5} = 1 - 0.2231 = 0.7769$

Example 5. Six coins are tossed 6400 times. Using the Poisson distribution, determine the approximate probability of getting six heads $x$ times.

(U.P.T.U. 2008)

Sol. Probability of getting one head with one coin = $\frac{1}{2}$.

- The probability of getting six heads with six coins = $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$
- Average number of six heads with six coins in 6400 throws = $np = 6400 \times \frac{1}{64} = 100$
- The mean of the Poisson distribution = 100

Approximate probability of getting six heads $x$ times when the distribution is Poisson

$= \frac{m^x e^{-m}}{x!} = \frac{(100)^x}{x!} e^{-100}$

EXERCISE 5.5

1. Fit a Poisson distribution to the following:
   $x$: 0 1 2 3 4
   $f$: 192 100 24 3 1

2. If the probability of a bad reaction from a certain insect is 0.001, determine the chance that out of 2000, individuals more than two will get a bad reaction.

3. If $X$ is a Poisson variate such that $P(X = 2) = 0.05P(X = 4)$ and $P(X = 6)$, find the standard deviation.

4. If a random variable has a Poisson distribution such that $P(1) = P(2)$, find
   (ii) $P(4)$.

5. Suppose that $X$ has a Poisson distribution. If $P(X = 2) = \frac{3}{5} P(X = 1)$ find, (i) $P(X = 0)$ (ii) $P(X = 3)$.

6. A certain screw making machine produces on average 2 defective screws out of 100, and packs them in boxes of 500. Find the probability that a box contains 15 defective screws.

(M.D.U. Dec. 2005)
7. A manufacturer knows that the condensers he makes contain an average 2% defective. He packs them in boxes of 100. What is the probability that a box selected at random will contain 3 or more defective condensers?

(M.D.U. May 2007)

8. Fit a Poisson distribution to the following and calculate theoretical frequencies:

\[
\begin{array}{ccc}
  x & 0 & 1 \\
  \hat{f} & 122 & 60 & 15 & 2 & 1
\end{array}
\]


9. Fit a Poisson distribution to the following data, given the number of yeast cells per square for 400 squares:

\[
\text{No. of cells per sq.}: 0 & 1 & 2 & 3 & 4 \\
\text{No. of squares}: 103 & 143 & 98 & 42 & 8 & 4 & 2 & 0 & 0 & 0
\]

(S.Y.T.U. 2007)

10. Show that in a Poisson distribution with unit mean, mean deviation about mean is \( \frac{2}{\sqrt{\pi}} \) times the standard deviation.

11. In a certain factory turning razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing no defective, one defective, and two defective blades respectively from a consignment of 10000 packets (K.U.K. Dec. 2006; Madras 2006 U.P.)

12. The probability that a man aged 35 years will die before reaching the age of 40 years may be taken as 0.018. Out of a group of 400 men, aged 35 years, what is the probability that 2 men will die within the next 5 years?

(P.T.U. Dec. 2005)

13. Suppose a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed throughout the book, what is the probability that 10 pages, selected at random, will be free from errors?

14. The side effects of a certain drug cause discomfort to only a few patients. The probability that any individual will suffer from these side effects is 0.02. If the drug is given to 500 patients, what is the probability that (i) exactly 2, (ii) 5 or more than 5 will suffer side effects?

(P.T.U. Dec. 2005)

15. The probability that a man aged 50 years will die within a year is 0.01125. What is the probability that out of 12 such men, at least 11 will reach their fifty-first birthday?

(Given \( e^{-0.1125} = 0.87371 \))

### Answers

1. \( 320 \times \frac{e^{-0.503} (0.503)^3}{r!} \)

2. 0.32

3. 3.1

4. (i) 2, (ii) \( \frac{2}{3e^2} \)

5. (i) \( e^{-\frac{2}{4}} \)

6. \( \frac{10^{15} e^{-10}}{(15)!} \)

7. 0.3236

8. 121.36 \times \frac{(0.5)^{r!}}{r!}

where \( r = 0, 1, 2, 3, 4 \)

9. Theoretical frequencies are 129, 61, 15, 3, 0 respectively

10. Theoretical frequencies are 129, 142, 52, 40, 13, 3, 1, 0, 0, 0

11. 58802, 196, 2

12. 0.01326

13. 0.4795

14. (i) 0.1784

(ii) 0.7150

15. 0.59166

### Normal Distribution


The normal distribution is a continuous distribution. It can be derived from the Binomial distribution in the limiting case when \( n \), the number of trials is very large and \( p \), the probability of a success, is close to \( \frac{1}{2} \). The general equation of the normal distribution is given by

\[
f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}
\]

where the variable \( x \) can assume all values from \(-\infty \) to \( +\infty \) and \( \mu \), \( \sigma \) called the parameters of the distribution, are respectively the mean and the standard deviation of the distribution and \( -\infty < \mu < +\infty, \sigma > 0 \). \( x \) is called the normal variate and \( f(x) \) is called probability density function of the normal distribution.

If a variable \( x \) has the normal distribution with mean \( \mu \) and standard deviation \( \sigma \), we briefly write \( x \sim N(\mu, \sigma^2) \).

The graph of the normal distribution is called the normal curve. It is bell-shaped and symmetrical about the mean \( \mu \). The two tails of the curve extend to \( +\infty \) and \( -\infty \) towards the positive and negative directions of the \( x \)-axis respectively and gradually approach the \( x \)-axis without ever meting it. The curve is unimodal and the mode of the normal distribution coincides with its mean \( \mu \). The line \( x = \mu \) divides the area under the normal curve above the \( x \)-axis into two equal parts. Thus, the median of the distribution also coincides with its mean and mode. The area under the normal curve between any two given ordinates \( x = x_1 \) and \( x = x_2 \) represents the probability of values falling into the given interval. The total area under the normal curve above the \( x \)-axis is 1.

5.27. BASIC PROPERTIES OF THE NORMAL DISTRIBUTION

The probability density function of the normal distribution is given by

\[
f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}
\]

1. The total area under normal probability curve is unity.

Normal probability curve is given by

\[
y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}
\]

Area under this curve

\[
\int_{-\infty}^{\infty} y \, dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx = 1
\]
4. The mode of the normal distribution.

The equation of the normal curve is
\[ y = \frac{N}{\sigma \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \]

Mode is the value of \( x \) corresponding to \( y = y_0 \), where \( y_0 \) is the maximum frequency. Proceeding as in Example 3, \( y \) is maximum when \( x = m \).

Hence the mode = the mean = \( m \).

5. The median of the normal distribution.

If \( M \) is the median of the normal distribution, we have
\[ \int_{-\infty}^{M} \frac{N}{\sigma \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx = \frac{N}{2} \]

\[ \int_{M}^{\infty} \frac{N}{\sigma \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx = \frac{1}{2} \]

\[ \int_{-\infty}^{M} e^{-\frac{t^2}{2\sigma^2}} \, dt + \int_{M}^{\infty} e^{-\frac{t^2}{2\sigma^2}} \, dt = \frac{1}{2} \]

Now
\[ \int_{-\infty}^{M} e^{-\frac{t^2}{2\sigma^2}} \, dt + \int_{M}^{\infty} e^{-\frac{t^2}{2\sigma^2}} \, dt = \frac{1}{\sqrt{\pi}} \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{M} e^{-t^2} \, dt + \frac{1}{\sqrt{\pi}} \int_{M}^{\infty} e^{-t^2} \, dt = \frac{1}{2} \]

\[ \int_{-\infty}^{M} e^{-\frac{t^2}{2\sigma^2}} \, dt = \frac{1}{2} \sqrt{\pi} = \frac{1}{2} \]

\[ \int_{M}^{\infty} e^{-\frac{t^2}{2\sigma^2}} \, dt = \frac{1}{2} \sqrt{\pi} = \frac{1}{2} \]

\[ \therefore \text{From (1)} \]

\[ \frac{1}{2} \int_{M}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx = \frac{1}{2} \]

\[ \int_{M}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx = 0 \Rightarrow M = m \]

Hence for the normal distribution, mean, median and mode coincide.

6. The variance and standard deviation of a normal distribution.

The equation of the normal curve is
\[ y = \frac{N}{\sigma \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}, \text{ Mean} = m \]

Variance
\[ \frac{1}{\sigma^2} \int_{-\infty}^{M} \frac{N}{\sigma \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{M} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx \]

Putting \( \frac{x-m}{\sigma \sqrt{2}} = t \), \( dt = \sigma \sqrt{2} \, dx \)

Hence \( y \), the ordinate is maximum when \( x = m \) i.e., the ordinate at the mean is the maximum ordinate.
Variance
\[ \frac{1}{\sigma^2} \int_{-\infty}^{\infty} 2a^2t^2 \cdot e^{-t^2} \cdot \sigma \sqrt{2 \pi} \, dt \]
\[ = \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^2 e^{-t^2} \, dt = \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} 2e^{-t^2} \, dt \]

Putting \( t = z, \ 2t \, dt = dz \) or
\[ dt = \frac{dz}{2\sqrt{z}} \]
\[ \therefore \text{Variance} = \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} z e^{-z^2} \, dz = \frac{2\sigma^2}{\sqrt{\pi}} \cdot 2\Gamma\left(\frac{3}{2}\right) = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \sigma^2. \]

Standard deviation = \( \sqrt{\text{Variance}} = \sigma \).

7. The points of inflexion of the normal curve.

Let the equation of the normal curve be
\[ y = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \]

Taking logarithms
\[ \log y = \log \frac{1}{\sigma \sqrt{2\pi}} - \frac{(x-m)^2}{2\sigma^2} \]

Differentiating w.r.t. \( x \)
\[ \frac{1}{y} \frac{dy}{dx} = \frac{x-m}{\sigma^2} \]

Differentiating again
\[ \frac{1}{y} \frac{d^2y}{dx^2} - \frac{1}{y^2} \left( \frac{dy}{dx} \right)^2 = -\frac{1}{\sigma^2} \]
\[ \frac{d^2y}{dx^2} = \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - \frac{x-m}{\sigma^2} \frac{y}{\sigma^2} = \frac{y}{\sigma^4} (x-m)^2 \]

At a point of inflexion,
\[ \frac{d^2y}{dx^2} = 0 \quad \therefore (x-m)^2 = \sigma^2 \]
\[ \therefore x-m = \pm \sigma \text{ or } x = m \pm \sigma. \]

Thus, the curve has two points of inflexion, one at \( m-\sigma \) and the other at \( m+\sigma \), i.e., a distance from the mean, equal to the standard deviation.

8. The mean deviation from the mean of the normal distribution is about \( \frac{\sigma}{3} \) of its standard deviation.

Let the equation of the normal curve be
\[ y = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx \]

Standard deviation = \( \sigma \).

Mean deviation from the mean
\[ = \int_{-\infty}^{\infty} y \, dx \cdot |x-m| \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx \]
\[ = \frac{\sigma^2}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} |x-m| e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx \]
\[ = \frac{\sigma}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} z \cdot e^{-\frac{z^2}{2\sigma^2}} \, dz \right] = \frac{\sigma}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2\sigma^2}} \, dz \right] \]
\[ = \frac{\sigma}{\sqrt{2\pi}} \left[ -\int_{-\infty}^{\infty} e^{-t^2} \, dt + \int_{0}^{\infty} e^{-t^2} \, dt \right] \]
\[ \therefore |z| = \begin{cases} -z & \text{if } z < 0 \\ z & \text{if } z \geq 0 \end{cases} \]

where \( t = z \)
\[ = \frac{\sigma}{\sqrt{2\pi}} \left[ \int_{0}^{\infty} e^{-z^2} \, dz + \int_{0}^{\infty} z e^{-\frac{z^2}{2\sigma^2}} \, dz \right] \]
\[ = \frac{\sigma}{\sqrt{2\pi}} \left[ \int_{0}^{\infty} e^{-z^2} \, dz + \int_{0}^{\infty} z e^{-\frac{z^2}{2\sigma^2}} \, dz \right] \]
\[ \therefore f(z) = \int_{0}^{\infty} f(x) \, dx = \int_{0}^{t} f(x) \, dx + \int_{t}^{\infty} f(x) \, dx = \int_{0}^{t} f(z) \, dz \]
\[ = \frac{\sigma}{\sqrt{2\pi}} \left[ \int_{0}^{\infty} e^{-z^2} \, dz \right] = \frac{\sigma}{\sqrt{2\pi}} \left[ \int_{0}^{t} e^{-t^2} \, dt \right] \]
\[ = \frac{\sigma}{\sqrt{2\pi}} \left[ -e^{-t^2} \right]_{0}^{t} = \frac{\sigma}{\sqrt{2\pi}} \left[ 0 - 1 \right] \]
\[ = \frac{\sigma}{\sqrt{2\pi}} \cdot 0.7979 \sigma = \frac{1}{3} \sigma \text{ (approx.)} \]
\[ = \frac{1}{3} \times \text{standard deviation (approx.)}. \]

5.28. STANDARD FORM OF THE NORMAL DISTRIBUTION

If \( X \) is a normal random variable with mean \( \mu \) and standard deviation \( \sigma \), then the random variable \( Z = \frac{X - \mu}{\sigma} \) has the normal distribution with mean 0 and standard deviation 1. The random variable \( Z \) is called the standardized (or standard) normal random variable.

The probability density function for the normal distribution in standard form is given by
\[ f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \]

It is free from any parameter. This helps us to compute areas under the normal probability curve by making use of standard tables.

Note 1. If \( f(x) \) is the probability density function for the normal distribution, then

\[ P(z_1 \leq Z \leq z_2) = \int_{z_1}^{z_2} f(z) \, dz = F(z_2) - F(z_1), \text{ where } F(z) = \int_{-\infty}^{z} f(t) \, dt = P(Z \leq z) \]

The function \( F(z) \) defined above is called the distribution function for the normal distribution.

Note 2. The probabilities \( P(z_1 \leq Z \leq z_2), P(z_1 < Z \leq z_2), P(z_1 < Z < z_2) \) and \( P(z_1 < Z < z_2) \) are all regarded to be the same.

Note 3. \( F(-z_1) = 1 - F(z_1) \).

**ILLUSTRATIVE EXAMPLES**

**Example 1.** A sample of 100 dry battery cells tested to find the length of life produced the following results:

- \( \bar{x} = 12 \) hours, \( \sigma = 3 \) hours.

Assuming the data to be normally distributed, what percentage of battery cells are expected to have a life:

(i) more than 15 hours
(ii) less than 6 hours
(iii) between 10 and 14 hours?

*Solution (P.T.U., May 2006)*

\[ z = \frac{x - \bar{x}}{\sigma} = \frac{x - 12}{3} \]

(i) When \( x = 15, z = 1 \)
\[ P(x > 15) = P(z > 1) = P(0 < z < 1) = 0.3413 = 0.3413 = 34.13\% \]

(ii) When \( x = 6, z = -2 \)
\[ P(x < 6) = P(z < -2) = P(0 < z < 1) = 0.4772 = 0.4772 = 47.72\% \]

(iii) When \( x = 10, z = \frac{2}{3} = 0.67 \)

When \( x = 14, z = \frac{2}{3} = 0.67 \)
\[ P(10 < x < 14) = P(-0.67 < z < 0.67) = 2 \times 0.2487 = 0.4974 = 49.74\% \]

**Example 2.** In a normal distribution, 31\% of the items are under 45 and 8\% are over 61.

Find the mean and standard deviation of the distribution.

*Solution (K.U.K., Dec. 2010)*

\[ \mu = \bar{x} = 45 \text{ and } \sigma = \sqrt{61 - 45} = 8 \]

\[ \mu = \bar{x} = 45 \text{ and } \sigma = \sqrt{61 - 45} = 8 \]

\[ P(z_1 < Z < z_2) = 0.31 \]

When \( x = 45, \) let \( z = z_1 \)
\[ P(z_1 < z < 0) = 0.31 \]

From the tables, the value of \( z \) corresponding to this area is 0.6
\[ z_1 = 0.5 \]

When \( x = 64, \) let \( z = z_2 \)
\[ P(0 < z < z_2) = 0.42 \]

From the tables, the value of \( z \) corresponding to this area is 1.4
\[ z_2 = 1.4 \]

Since
\[ z = \frac{x - \bar{x}}{\sigma} \]
\[ -0.5 = \frac{45 - \bar{x}}{3} \]
and
\[ 1.4 = \frac{64 - \bar{x}}{3} \]

Subtracting
\[ -0.5 = -1.9 \sigma \]
\[ \sigma = 10 \]

From (1),
\[ 45 - \bar{x} = 0.5 \times 10 = 5 \]
\[ \bar{x} = 50 \]

**EXERCISE 5.5**

1. If \( z \) is the standard normal variate, then find the following probabilities:
   (i) \( P(1 \leq z \leq 2) \)
   (ii) \( P(-2.3 \leq z \leq -1.5) \)
   (iii) \( P(-0.42 \leq z \leq 0.2) \)
   (iv) \( P(z \leq 1) \)
   (v) \( P(1 \geq z) \)

2. Let \( X \) be a random variable having a normal distribution with mean 30 and standard deviation 5. Find the probability that
   (i) \( 25 < x < 40 \)
   (ii) \( x < 30 \)
   (iii) \( x > 21.8 \)

3. The mean height of 500 male students in a certain college is 151 cm and the standard deviation is 15 cm. Assuming the heights are normally distributed, find how many students have heights between 120 and 155 cm?

4. An aptitude test for selecting officers in a bank is conducted on 1000 candidates. The average score is 42 and the standard deviation of score is 24. Assuming normal distribution for the scores, find
   (i) the number of candidates whose scores exceed 60
   (ii) the number of candidates whose scores lie between 30 and 60
   (iii) the number of candidates whose scores exceed 80

5. If the mean height of an Indian police inspector be 170 cm with variance 25 cm², how many inspectors out of 1000 would you expect
   (i) between 170 cm and 180 cm
   (ii) less than 160 cm?
7. The income of a group of 10,000 persons was found to be normally distributed with mean = ₹ 70, p.m. and standard deviation = ₹ 50. Show that of this group about 95% had income exceeding ₹ 658 and only 5% had income exceeding ₹ 832. What was the lowest income among the richest 1000?

8. In a normal distribution, 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution?

9. Let X denote the number of scores on a test. If X is normally distributed with mean 100 and standard deviation 15, find the probability that X does not exceed 130.

10. It is known from the past experience that the number of telephone calls made daily in a certain community between 3 p.m. and 4 p.m. has a mean of 352 and a standard deviation of 31. What percentage of the time will there be more than 400 telephone calls made in this community between 3 p.m. and 4 p.m.?

11. Students of a class were given a mechanical aptitude test. Their marks were found to be normally distributed with mean 60 and standard deviation 5. What per cent of students score (i) more than 60 marks? (ii) less than 56 marks?

12. In an examination taken by 500 candidates, the average and the standard deviation of marks obtained (normally distributed) are 45% and 10%. Find approximately (i) how many will pass if 50% is fixed as a minimum? (ii) what should be the minimum if 350 candidates are to pass?

13. In a distribution, exactly normal, 9.85% of the items are under 40 and 89.97% are under 50. What are the mean and standard deviation of the distribution?

14. The income distribution of workers in a certain factory was found to be normal with mean Rs. 500 and standard deviation of Rs. 50. There were 228 workers getting above Rs. 650. How many workers were there in all?

15. The mean inside diameter of a sample of 500 washers produced by a machine is 5.02 mm and the standard deviation is 0.05 mm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 4.80 to 5.08 mm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed.

16. Fit a normal curve to the following distribution:

<table>
<thead>
<tr>
<th>x</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>


**Answers**

1. (i) 0.1359 (ii) 0.0561 (iii) 0.064
2. (i) 0.2877 (ii) 0.6256 (iii) 0.3174
3. (i) 0.7653 (ii) 0.3174 (iii) 0.7231 (iv) 0.0013
4. 300 (i) 252 (ii) 533
5. 477 (i) 23 (ii) 23
6. Rs. 506.50 (8. μ = 50.3, σ = 10.33)
7. 9.9772 (i) 6.06%
8. 50% (ii) 21.2% (iii) 84%
9. 79 (i) 79 (ii) 35%
10. μ = 157.29, σ = 54.05 (i) 11 (ii) 11 (iii) 11
11. 10,000 (i) 10,000 (ii) 10,000 (iii) 10,000
12. \( y = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \mu)^2}{2\sigma^2}} \)
CHAPTER 6

Hypothesis Testing

SAMPLING AND TESTS OF SIGNIFICANCE

6.1. POPULATION OR UNIVERSE

An aggregate of objects (animate or inanimate) under study is called population or universe. It is thus a collection of individuals or of their attributes (qualities) or of results of operations which can be numerically specified.

A universe containing a finite number of individuals or members is called a finite universe. For example, the universe of the weights of students in a particular class or the universe of smokes in Rohtak district.

A universe with infinite number of members is known as an infinite universe. For example, the universe of pressures at various points in the atmosphere.

In some cases, we may be even ignorant whether or not a particular universe is infinite, e.g., the universe of stars.

The universe of concrete objects is an existent universe. The collection of all possible ways in which a specified event can happen is called a hypothetical universe. The universe of heads and tails obtained by tossing a coin an infinite number of times (provided that it does not wear out) is a hypothetical one.

6.2. SAMPLING

The statistician is often confronted with the problem of discussing universe of which he cannot examine every member i.e., of which complete enumeration is impracticable. For example, if we want to have an idea of the average per capita income of the people of India, enumeration of every earning individual in the country is a very difficult task. Naturally, the question arises: What can be said about a universe of which we can examine only a limited number of members? This question is the origin of the Theory of Sampling.

A finite sub-set of a universe is called a sample. A sample is thus a small portion of the universe. The number of individuals in a sample is called the sample size. The process of selecting a sample from a universe is called sampling.

The theory of sampling is a study of relationship existing between a population and samples drawn from the population. The fundamental object of sampling is to get as much information as possible of the whole universe by examining only a part of it. An attempt is thus made through sampling to give the maximum information about the parent universe with the minimum effort.
Sampling is quite often used in our day-to-day practical life. For example, in a shop we assess the quality of sugar, rice or any other commodity by taking only a handful of it from the bag and then decide whether to purchase it or not. A housewife normally tests the cooked products to find if they are properly cooked and contain the proper quantity of salt or sugar, by taking a spoonful of it.

6.3. PARAMETERS OF STATISTICS

The statistical constants of the population such as mean, the variance etc., are known as the parameters. The statistical concepts of the sample from the members of the sample to estimate the parameters of the population from which the sample has been drawn are known as statistic.

Population mean and variance are denoted by μ and σ², while those of the sample are given by \( \bar{x} \), \( s^2 \).

6.4. STANDARD ERROR (S.E.)

The standard deviation of the sampling distribution of a statistic is known as the standard error (S.E.).

It plays an important role in the theory of large samples and it forms a basis of the testing of hypothesis. If \( t \) is any statistic, for large sample,

\[ z = \frac{t - \mu}{\sigma(t)} \]

is normally distributed with mean 0 and variance unity.

For large sample, the standard errors of some of the well known statistic are listed below:

- \( n \) — sample size
- \( \sigma^2 \) — population variance
- \( s^2 \) — sample variance
- \( p \) — population proportion
- \( Q = 1 - p \)
- \( n_1, n_2 \) — are sizes of two independent random sample.

<table>
<thead>
<tr>
<th>No.</th>
<th>Statistic</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \bar{x} )</td>
<td>( \sigma/\sqrt{n} )</td>
</tr>
<tr>
<td>2.</td>
<td>( s )</td>
<td>( \sigma/\sqrt{2n} )</td>
</tr>
<tr>
<td>3.</td>
<td>Difference of two sample means ( \bar{x}_1 - \bar{x}_2 )</td>
<td>( \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} )</td>
</tr>
<tr>
<td>4.</td>
<td>Difference of two sample standard deviation ( s_1 - s_2 )</td>
<td>( \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} )</td>
</tr>
<tr>
<td>5.</td>
<td>Difference of two sample proportions ( p_1 - p_2 )</td>
<td>( \sqrt{\frac{Q_1}{n_1} + \frac{Q_2}{n_2}} )</td>
</tr>
<tr>
<td>6.</td>
<td>Observed sample proportion ( p )</td>
<td>( \sqrt{\frac{Q}{n}} )</td>
</tr>
</tbody>
</table>

6.5. TEST OF SIGNIFICANCE

(P.T.U. 2008)

An important aspect of the sampling theory is to study the test of significance. Which will enable us to decide, on the basis of the results of the sample, whether

HYPOTHESIS TESTING

(i) the deviation between the observed sample statistic and the hypothetical parameter value or
(ii) the deviation between two sample statistics is significant or might be attributed due to chance or the fluctuations of the sampling.

For applying the tests of significance, we first set up a hypothesis which is a definite statement about the population parameter called Null hypothesis denoted by \( H_0 \).

Any hypothesis which is complementary to the null hypothesis \( H_0 \) is called an Alternative hypothesis denoted by \( H_1 \).

For example if we want to test the null hypothesis that the population has a specified mean \( \mu_0 \), then we have

\[ H_0: \mu = \mu_0 \]

Alternative hypothesis will be

(i) \( H_1: \mu > \mu_0 \) (two tailed alternative hypothesis).
(ii) \( H_1: \mu > \mu_0 \) (right tailed alternative hypothesis (or) single tailed).
(iii) \( H_1: \mu < \mu_0 \) (left tailed alternative hypothesis (or) single tailed).

Hence alternative hypothesis helps to know whether the test is two tailed test or one tailed test.

6.6. CRITICAL REGION

A region corresponding to a statistic \( t \), in the sample space \( S \) which amounts to rejection of the null hypothesis \( H_0 \) is called as critical region or region of rejection. The region of the sample space \( S \) which amounts to the acceptance of \( H_0 \) is called acceptance region.

6.7. LEVEL OF SIGNIFICANCE

The probability of the value of the variate falling in the critical region is known as level of significance.

The probability \( \alpha \) that a random value of the statistic \( t \) belongs to the critical region is known as the level of significance.

\[ P(t \in \omega | H_0) = \alpha \]

i.e., the level of significance is the size of the type I error or the maximum producer's risk.

6.8. ERRORS IN SAMPLING

The main aim of the sampling theory is to draw a valid conclusion about the population parameters. On the basis of the sample results. In doing this we may commit the following two types of errors:

Type I Error. When \( H_0 \) is true, we may reject it.

\[ P(\text{Reject } H_0 \text{ when it is true}) = P(\text{Reject } H_0/H_0) = \alpha \]

\( \alpha \) is called the size of the type I error also referred to as producer's risk.

Type II Error. When \( H_0 \) is wrong we may accept it if \( P(\text{Accept } H_0 \text{ when it is wrong}) = P(\text{Accept } H_0/H_0) = \beta \)

\( \beta \) is called the size of the type II error, also referred to as consumer's risk.
Critical values or significant values

The values of the test statistic which separates the critical region and acceptance region is called the critical values or significant value.

This value is dependent on (i) the level of significance used and (ii) the alternative hypothesis, whether it is one tailed or two tailed.

For larger samples corresponding to the statistic $t$, the variable $z = \frac{t - E(t)}{S.E.(t)}$ is normally distributed with mean 0 and variance 1. The value of $z$ given above under the null hypothesis is known as test statistic.

The critical value of $z$ at level of significance $\alpha$ for a two tailed test is given by

$$p( | z | > z_\alpha ) = \alpha$$

i.e., $z_\alpha$ is the value of $z$ so that the total area of the critical region on both tails is $\alpha$. Since the normal curve is a symmetrical, from equation (1), we get

$$p(z > z_\alpha) = \frac{1}{2} \alpha; p(z < -z_\alpha) = \frac{1}{2} \alpha; p(z > z_\alpha) = \alpha/2$$

i.e., the area of each tail is $\alpha/2$.

The critical value $z_\alpha$ is that value such that the area to the right of $z_\alpha$ is $\alpha/2$ and the area to the left of $-z_\alpha$ is $\alpha/2$.

In the case of one tailed test:

$$p(z > z_\alpha) = \alpha$$ if it is right tailed; $p(z < -z_\alpha) = \alpha$ if it is left tailed.

The critical value of $z$ for a single—tailed test (right or left) at level of significance $\alpha$ is same as the critical value of $z$ for two-tailed test at level of significance $2\alpha$.

Using the equation, also using the normal tables, the critical value of $z$ at different level of significance ($\alpha$) for both single tailed and two tailed test are calculated and listed below. The equations are

$$p( | z | > z_\alpha ) = \alpha; p(z > z_\alpha) = \alpha; p(z < -z_\alpha) = \alpha$$

### HYPOTHESIS TESTING

<table>
<thead>
<tr>
<th>Level of significance</th>
<th>1% (0.01)</th>
<th>5% (0.05)</th>
<th>10% (0.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two tailed test</td>
<td>$z_{1.0}$ = 2.58</td>
<td>$z_{1.0}$ = 1.966</td>
<td>$z_{1.0}$ = 1.645</td>
</tr>
<tr>
<td>Right tailed</td>
<td>$z_{0.5}$ = 2.33</td>
<td>$z_{0.5}$ = 1.645</td>
<td>$z_{0.5}$ = 1.28</td>
</tr>
<tr>
<td>Left tailed</td>
<td>$z_{0.5}$ = 2.33</td>
<td>$z_{0.5}$ = 1.645</td>
<td>$z_{0.5}$ = 1.28</td>
</tr>
</tbody>
</table>

Note. The following steps may be adopted in testing of statistical hypothesis:

Step 1: Null hypothesis. Set up $H_0$ is clear terms.

Step 2: Alternative hypothesis. Set up $H_1$, so that we could decide whether we should use one-tailed test or two tailed test.

Step 3: Level of significance. Select the appropriate level of significance in advance depending on the reliability of the estimates.

Step 4: Test statistic. Compute the test statistic $z = \frac{t - E(t)}{S.E.(t)}$ under the null hypothesis.

Step 5: Conclusion. Compare the computed value of $z$ with the critical value $z_\alpha$ at level of significance ($\alpha$).

If $| z | > z_\alpha$, we reject $H_0$ and conclude that there is significant difference. If $| z | < z_\alpha$, we accept $H_0$ and conclude that there is no significant difference.

**TEST OF SIGNIFICANCE FOR LARGE SAMPLES**

If the sample size $n > 30$, the sample is taken as large sample. For such sample we apply normal test, as Binomial, Poisson, chisquare etc. are closely approximated by normal distributions assuming the population as normal.

Under large sample test, the following are the important tests to test the significance.

1. Testing of significance for single proportion.
2. Testing of significance for difference of proportions.
3. Testing of significance for single mean.
4. Testing of significance for difference of means.

### 6.9. TESTING OF SIGNIFICANCE FOR SINGLE PROPORTION

This test is used to find the significant difference between proportion of the sample and the population.

Let $X$ be the number of successes in $n$ independent trials with constant probability $P$ of success for each trial.

$$E(X) = np; \ V(X) = npq; \ Q = 1 - p = \text{Probability of failure}.$$  

Let $p = X/n$ called the observed proportion of success.

$$E(p) = (E(X)/n) = \frac{1}{n} E(X) = \frac{n}{n} = P; \ E(p) = P$$

$$V(p) = \ V(X/n) = \frac{1}{n^2} \ v(X) = \frac{npq}{n^2} = pq/n$$
HYPOTHESIS TESTING

The test statistic

\[ z = \frac{p - \bar{p}}{\sqrt{\frac{PQ}{n}}} \]

where \( \bar{p} = \frac{36}{100} = 0.36 \)

\[ z = \frac{0.36 - 0.33}{\sqrt{\frac{1 \times 2}{3 \times 3 \times 1000}}} = 0.03496 \]

\[ |z| = 0.03496 < 1.96 \]

Conclusion. Accept the hypothesis.

As \( |z| < z_{0.05} \), the tabulated value of \( z \) at 5% level of significance.

\( H_0 \) is accepted, we conclude that the die is unbiased.

To find 95% confidence limits of the proportion.

It is given by

\[ P \pm z_{0.05} \sqrt{\frac{PQ}{n}} \]

i.e.,

\[ 0.33 \pm 0.0097 = 0.33 \pm 0.0097 = 0.3203 \text{ and } 0.3397. \]

Example 3. A manufacturer claims that only 4% of his products supplied by him are defective. A random sample of 600 products contained 36 defectives. Test the claim of the manufacturer.

Sol. (i) \( P = \) observed proportion of success.

i.e.,

\[ P = \frac{36}{600} = 0.06 \]

\[ p = \text{proportion of defectives in the population} = 0.04 \]

\( H_0: p = 0.04 \) is true.

\( H_0, p = 0.04 \) is not rejected (two tailed test)

(iii) If we want to reject, only if \( p > 0.04 \) then (right tailed).

Under \( H_0 \),

\[ z = \frac{P - \bar{p}}{\sqrt{\frac{PQ}{n}}} = \frac{0.06 - 0.04}{\sqrt{\frac{0.04 \times 0.96}{600}}} = 2.5. \]

Conclusion. Since \( |z| = 2.5 > 1.96 \), we reject the hypothesis \( H_0 \) at 5% level of significance.

If \( H_0 \) is taken as \( p > 0.04 \) we apply right tailed test.

\[ |z| = 2.5 > 1.645 (z_{0.05}) \text{ we reject the null hypothesis here also.} \]

Example 4. A machine is producing bolts of which a certain fraction is defective. A random sample of 400 is taken from a large batch and is found to contain 30 defective bolts. Does this indicate that the proportion of defectives is larger than that claimed by the manufacturer where the manufacturer claims that only 5% of his product are defective. Find 95% confidence limits of the proportion of defective bolts in batch.

Sol. Null hypothesis. \( H_0: P = \frac{5}{100} = 0.05 \)

Alternative hypothesis. \( P > 0.05 \) (Right tailed test).
\[ p = \text{observed proportion of sample} = \frac{30}{400} = 0.075 \]

Under \( H_0 \), the test statistic \( z = \frac{p - P}{\sqrt{PQ/n}} \) gives \[ z = \frac{0.075 - 0.05}{\sqrt{0.05 \times 0.95 \times \frac{1}{400}}} = 2.2941. \]

**Conclusion.** The tabulated value of \( z \) at 5% level of significance for right-tailed test is \( z = 1.645 \). Since \( |z| = 2.2941 > 1.645 \), \( H_0 \) is rejected at 5% level of significance, i.e., the proportion of defective is larger than the manufacturer claim.

To find 95% confidence limits of the proportion,

It is given by \( P \pm z_{a/2} \sqrt{PQ/n} \)

\[ 0.05 \pm 1.96 \sqrt{\frac{0.05 \times 0.95}{400}} = 0.05 \pm 0.02135 = 0.02865 \text{ and 0.07135}. \]

Hence 95% confidence limits for the proportion of defective bolts are (0.07135, 0.02865).

**Example 6.** A bag contains defective articles, the exact number of which is not known. A sample of 100 from the bag gives 10 defective articles. Find the limits for the proportion of defective articles in the bag.

Sol. Here \( P = \) proportion of defective articles = \( \frac{10}{100} = 0.1 \); \( q = 1 - p = 1 - 0.1 = 0.9 \), and the level of significance is 5% \( z_{a/2} = 1.96 \).

Also the proportion of population \( P \) is not given. To get the confidence limit, we use \( P \approx q \) and it is given by \( P \pm z_{a/2} \sqrt{pq/n} \)

\[ 0.1 \pm 1.96 \sqrt{\frac{0.1 \times 0.9}{100}} = 0.1 \pm 0.0588 = 0.0412 \text{ and 0.1588}. \]

Hence 95% confidence limits for defective article in the bag are (0.0412, 0.1588).

**EXERCISE 6.1**

1. A sample of size 600 persons selected at random from a large city shows that the percentage of males in the sample is 53. It is believed that the ratio of males to the total population in the city is 56. Test whether the belief is confirmed by the observation.

2. In a city a sample of 1000 people were taken and out of them 549 are vegetarian and the rest non-vegetarian. Can we say that the both habits of eating (Vegetarian or non-vegetarian) are equally popular in the city? Are the data at (i) 1% level of significance (ii) 5% level of significance?

3. 325 men out of 600 men chosen from a big city were found to be smokers. Does this information support the conclusion that the majority of men in the city are smokers.

4. A random sample of 500 bolts was taken from a large consignment and 65 were found to be defective. Find the percentage of defective bolts in the consignment.

5. In a hospital 475 female and 825 male babies were born in a week. Do these figures confirm hypothesis that males and females are born in equal number?

6. 400 apples are taken at random from a large basket and 40 are found to be bad. Estimate the proportion of bad apples in the basket and assign limits within which the percentage must probably lie.

**HYPOTHESIS TESTING**

7. A manufacturer claims that only 10% of the articles produced are below the standard quality. Out of a random inspection of 300 articles, 37 are found to be of poor quality. Test the manufacturer's claim at 5% level of significance.

**Answers**

1. \( H_0 \) accepted at 5% level
2. \( H_0 \) rejected at 5% level, accepted at 1% level
3. \( H_0 \) rejected at 5% level
4. Between 17.51 and 8.49
5. \( H_0 \) accepted at 5% level
6. \( \beta = 11.5 \)
7. \( H_0 \) accepted at 5% level.

**6.10. TEST OF DIFFERENCE BETWEEN PROPORTIONS**

Consider two samples \( X_1 \) and \( X_2 \) of sizes \( n_1 \) and \( n_2 \), respectively taken from two different populations. To test the significance of the difference between the sample proportion \( p_1 \) and \( p_2 \) by the test statistic under the null hypothesis \( H_0 \), that there is no significant difference between the two sample proportion, we have

\[ z = \frac{p_1 - p_2}{\sqrt{pq / (n_1 + n_2)}}, \]

where \( P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} \) and \( Q = 1 - P \).

**ILLUSTRATIVE EXAMPLES**

**Example 1.** Before an increase in excise duty on tea, 800 people out of a sample of 1000 persons were found to be tea drinkers. After an increase in the duty, 800 persons were known to be tea drinkers in a sample of 1200 people. Do you think that there has been a significant decrease in the consumption of tea after the increase in the excise duty?

Sol. Here \( n_1 = 800, n_2 = 1200 \)

\[ p_1 = \frac{X_1}{n_1} = \frac{800}{1000} = 0.8, \quad p_2 = \frac{X_2}{n_2} = \frac{800}{1200} = 0.666 \]

\[ p = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2} = \frac{0.8 \times 1000 + 0.666 \times 1200}{1000 + 1200} = 0.7 \]

Null hypothesis \( H_0: p_1 = p_2 \) i.e., there is no significant difference in the consumption of tea before and after increase of excise duty.

\( H_1: p_1 > p_2 \) (right tailed test)

The test statistic \( z = \frac{p_1 - p_2}{\sqrt{p(1-p) / (n_1 + n_2)}} \) gives \( z = 0.8 - 0.666 = 0.134 \).

Conclusion. Since the calculated value of \( |z| = 1.645 \) also \( |z| = 2.33 \), both the significant value of \( z \) at 5% and 1% level of significance. Hence \( H_0 \) is rejected i.e., there is a significant decrease in the consumption of tea due to increase in excise duty.

**Example 2.** A machine produced 16 defective articles in a batch of 500. After overhauling it produced 3 defective articles in a batch of 100. Has the machine improved?

(M.D.U. May 2007)
**Solution**

\( p_1 = \frac{16}{500} = 0.032; n_1 = 500 \)

\( p_2 = \frac{3}{100} = 0.03; n_2 = 100 \)

**Null hypothesis** \( H_0 \): The machine has not improved due to overheating, \( p_1 = p_2 \).

\( H_1: p_1 > p_2 \) (right tailed) \( \therefore P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2} = \frac{19}{600} = 0.032 \)

Under \( H_0 \), the test statistic

\[
 z = \frac{p_1 - p_2}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} = \frac{0.032 - 0.03}{\sqrt{\frac{0.032(0.968)}{500} + \frac{0.03(0.975)}{100}}} = 0.104. 
\]

**Conclusion.** The calculated value of \( |z| < 1.645 \), the significant value of \( z \) at 5% level of significance. \( H_0 \) is accepted, i.e., the machine has not improved due to overheating.

**Example 3.** In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations.

(M.D.U. Dec. 2010)

\( p_1 = 0.3; n_1 = 1200; p_2 = 0.25; n_2 = 900 \)

**H_0:** Sample proportions are equal i.e., the difference in population proportions is likely to be hidden in sampling.

\[
 H_1: p_1 \neq p_2 \]

\[
 z = \frac{P_1 - P_2}{\sqrt{\frac{P_1 Q_1 n_1}{n_1} + \frac{P_2 Q_2 n_2}{n_2}}} = \frac{0.3 - 0.25}{\sqrt{\frac{0.3(0.7)(1200)}{1200} + \frac{0.25(0.75)(900)}{900}}} = 2.5376. 
\]

**Conclusion.** Since \(|z| = 1.96\), the significant value of \( z \) at 5% level of significance, \( H_0 \) is rejected. However \(|z| = 2.58\), the significant value of \( z \) at 1% level of significance, \( H_0 \) is accepted. At 5% level these samples will reveal the difference in the population proportions.

**Example 4.** 600 articles from a factory are examined and found to be 2% defective. 800 similar articles from a second factory are found to have only 1.5% defective. Can it reasonably be concluded that the product of the first factory are inferior to those of second?

\( n_1 = 600; n_2 = 800 \)

\( p_1 = \) proportion of defective from first factory = 2% = 0.02

\( p_2 = \) proportion of defective from second factory = 1.5% = 0.015

**H_0:** There is no significant difference between the two products i.e., the products do not differ in quality.

\( H_1: p_1 < p_2 \) (one tailed test)

Under \( H_0 \),

\[
 z = \frac{P_1 - P_2}{\sqrt{\frac{P_1 Q_1}{n_1} + \frac{P_2 Q_2}{n_2}}} = \frac{0.02(600) + (0.015)(800)}{500 + 800} = 0.01692; Q = 1 - P = 0.9830. 
\]

**Exercise 6.2**

1. Random sample of 400 men and 600 women were asked whether they would have a school near their residence. 200 men and 320 women were in favour of proposal. Test the hypothesis that the proportion of men and women in favour of the proposal are same at 5% level of significance.

2. In a town A, there were 956 births of which 52.5% was males while in towns A and B combined, this proportion in total of 1406 births was 0.496. Is there any significant difference in the proportion of male births in the two towns?

3. In a referendum submitted to the students body at a university, 850 men and 660 women voted. 500 men and 320 women voted yes. Does this indicate a significant difference of opinion between men and women on this matter at 1% level?

4. A manufacturing firm claims that its brand A product outsells its brand B product by 8%. If it is found that 42 out of a sample of 200 person prefer brand A and 18 out of another sample of 100 person prefer brand B. Test whether the 8% difference is a valid claim.

**Answers**

1. \( H_0 \) accepted

2. \( H_0 \) rejected

3. \( H_0 \) accepted

4. \( H_0 \) accepted

**6.11 TEST OF SIGNIFICANCE FOR SINGLE MEAN**

To test whether the difference between sample mean and population mean is significant or not.

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from a large population \( X_1, X_2, \ldots, X_N \) of size \( N \) with mean \( \mu \) and variance \( \sigma^2 \). \( \therefore \) the standard error of mean of a random sample of size \( n \) from a population with variance \( \sigma^2 \) is \( \sigma/\sqrt{n} \).

To test whether the given sample of size \( n \) has been drawn from a population with mean \( \mu \) i.e., to test whether the difference between the sample mean and population mean is significant or not. Under the null hypothesis that there is no difference between the sample mean and population mean.

The test statistic is \( z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \), where \( \sigma \) is the standard deviation of the population.

If \( \sigma \) is not known, we use the test statistic \( z = \frac{\bar{X} - \mu}{s/\sqrt{n}} \), where \( s \) is the standard deviation of the sample.

**Note.** If the level of significance is \( \alpha \) and \( z_\alpha \) is the critical value then \( z < -z_\alpha < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \) or \( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z < z_\alpha \).
ILLUSTRATIVE EXAMPLES

Example 1. A normal population has a mean of 6.8 and standard deviation of 1.5. A sample of 400 members gave a mean of 6.75. Is the difference significant?

Sol. H₀: There is no significant difference between \( \bar{x} \) and \( \mu \).

H₁: There is significant difference between \( \bar{x} \) and \( \mu \).

Given \( \mu = 6.8 \), \( \sigma = 1.5 \), \( \bar{x} = 6.75 \) and \( n = 400 \)

\[
\frac{|z|}{\sqrt{n}} = \frac{| \bar{x} - \mu |}{\sigma/\sqrt{n}} = \frac{6.75 - 6.8}{1.5/\sqrt{400}} = | -0.67 | = 0.67
\]

Conclusion. As the calculated value of \( |z| < z_\alpha = 1.96 \) at 5% level of significance, \( H_0 \) is accepted i.e., there is no significant difference between \( \bar{x} \) and \( \mu \).

Example 2. A random sample of 900 members has a mean 3.4 cm. Can it be reasonably regarded as a sample from a large population of mean 3.2 cm and S.D. 2.3 cm?

Sol. Here \( n = 900 \), \( \bar{x} = 3.4 \), \( \mu = 3.2 \), \( \sigma = 2.3 \)

H₀: Assume that the sample is drawn from a large population with mean 3.2 and S.D. 2.3

\[ H₀: \mu = 3.25 \text{ (Apply two tailed test)} \]

Under \( H₀ \), \[ \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{3.4 - 3.2}{2.3/\sqrt{900}} = 0.261 \]

Conclusion. As the calculated value of \( |z| = 0.261 < 1.96 \) the significant value of z at 5% level of significance, \( H₀ \) is accepted i.e., the sample is drawn from the population with mean 3.2 and S.D. 2.3.

Example 3. The mean weight obtained from a random sample of size 100 is 64 g. The S.D. of the weight distribution of the population is 3 g. Test the statement that the mean weight of the population is 67 g at 5% level of significance. Also set up 95% confidence limits of the mean weight of the population.

Sol. Here \( n = 100 \), \( \mu = 67 \), \( \bar{x} = 64 \), \( \sigma = 3 \)

H₀: There is no significant difference between sample and population mean.

i.e., \( \mu = 67 \), the sample is drawn from the population with \( \mu = 67 \)

\[ H₀: \mu = 67 \text{ (Two tailed test)} \]

Under \( H₀ \), \[ \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{64 - 67}{3/\sqrt{100}} = -10 \Rightarrow |z| = 10. \]

Conclusion. Since the calculated value of \( |z| > 1.96 \), the significant value of z at 5% level of significance, \( H₀ \) is rejected i.e., the sample is not drawn from the population with mean 67.

HYPOTHESIS TESTING

To find 95% confidence limits. It is given by \( \bar{x} \pm z \frac{\sigma}{\sqrt{n}} = 64 \pm 2.58 \frac{3}{\sqrt{100}} = 64.774 \).

Example 4. The average marks in Mathematics of a sample of 100 students was 51 with a S.D. of 6 marks. Could this have been a random sample from a population with average marks 50?

Sol. Here \( n = 100 \), \( \bar{x} = 51 \), \( s = 6 \), \( \mu = 50 \), \( \sigma \) is unknown.

H₀: The sample is drawn from a population with mean 50, \( \mu = 50 \)

Under \( H₀ \), \[ z = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{51 - 50}{6/\sqrt{100}} = 1.6666 \]

Conclusion. Since \( |z| < 1.96 \), \( z_\alpha \), the significant value of z at 5% level of significance, \( H₀ \) is accepted i.e., the sample is drawn from the population with mean 50.

EXERCISE 6.3

1. A sample of 1000 students from a university was taken and their average weight was found to be 120 pounds with a S.D. of 20 pounds. Could the mean weight of students in the population be 120 pounds?
2. A sample of 400 male students is found to have a mean height of 160 cm. Can it be reasonably regarded as a sample from a large population with mean height 162.5 cm and standard deviation 4.5 cm?
3. A random sample of 200 measurements from a large population gave a mean value of 50 and a S.D. of 9. Determine 95% confidence interval for the mean of population.
4. The guaranteed average life of a certain type of bulb is 1000 hours with a S.D. of 125 hours. It is decided to sample the output so as to ensure that 90% of the bulbs do not fall short of the guaranteed average by more than 2.5%. What must be the minimum size of the sample.

6. The heights of college students in a city are normally distributed with S.D. 6 cm. A sample of 1000 students has mean height 158 cm. Test the hypothesis that the mean height of college students in the city is 160 cm.

Answers

1. \( H_0 \) rejected
2. \( H_0 \) accepted
3. 48.8 and 51.2
4. \( n = 4 \)
5. \( H_0 \) rejected both at 1% to 5% level of significance.

6.12. TEST OF SIGNIFICANCE FOR DIFFERENCE OF MEANS OF TWO LARGE SAMPLES

Let \( \bar{x}_1 \) be the mean of a sample of size \( n_1 \) from a population with mean \( \mu_1 \) and variance \( \sigma_1^2 \). Let \( \bar{x}_2 \) be the mean of an independent sample of size \( n_2 \) from another population with mean \( \mu_2 \) and variance \( \sigma_2^2 \). The test statistic is given by \( z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \)

Under the null hypothesis that the samples are drawn from the same population where

\[ \sigma_1 = \sigma_2 = \sigma \text{ i.e., } \mu_1 = \mu_2 \text{ the test statistic is given by } z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \]
Note 1. If \( \sigma_1, \sigma_2 \) are not known and \( \sigma_1 \neq \sigma_2 \) the test statistic in this case is:
\[
z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}
\]

Note 2. If \( \sigma \) is not known and \( \sigma = \sigma_0 \)
We use \( \sigma^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2} \) to calculate \( z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \)

### ILLUSTRATIVE EXAMPLES

Example 1. The average income of persons was Rs 210 with a S.D. of Rs 10 in sample of 100 people of a city. For another sample of 150 persons, the average income was Rs 220 with S.D. of Rs 12. The S.D. of incomes of the people of the city was Rs 11. Test whether there is any significant difference between the average incomes of the localities.

**Sol.** Here \( n_1 = 100, n_2 = 150, \bar{x}_1 = 210, \bar{x}_2 = 220, s_1 = 10, s_2 = 12 \).

**Null hypothesis.** The difference is not significant, i.e., there is no difference between the incomes of the localities.

**H_0:** \( \bar{x}_1 = \bar{x}_2, H_1: \bar{x}_1 \neq \bar{x}_2 \)

**Under H_0:**
\[
z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{210 - 220}{\sqrt{\frac{10^2}{100} + \frac{12^2}{150}}} = -7.428 \therefore |z| = 7.1428.
\]

**Conclusion.** As the calculated value of \( |z| > 1.96 \) the significant value of \( z \) at 5% level of significance, \( H_0 \) is rejected, i.e., there is significant difference between the average incomes of the localities.

Example 2. Intentional tests were given to two groups of boys and girls.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>S.D.</th>
<th>size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Girls</td>
<td>75</td>
<td>8</td>
<td>60</td>
</tr>
<tr>
<td>Boys</td>
<td>73</td>
<td>10</td>
<td>100</td>
</tr>
</tbody>
</table>

Examine if the difference between mean scores is significant. (M.D.U. Dec. 2011)

**Sol. Null hypothesis H_0:** There is no significant difference between mean scores i.e., \( \bar{x}_1 = \bar{x}_2 \)

**Under the null hypothesis:**
\[
z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{75 - 73}{\sqrt{\frac{10^2}{60} + \frac{10^2}{100}}} = 1.3912.
\]

**Conclusion.** As the calculated value of \( |z| < 1.96 \), the significant value of \( z \) at 5% level of significance, \( H_0 \) is accepted, i.e., there is no significant difference between mean scores.

Example 3. For sample I, \( n_1 = 1000, \Sigma x = 49,000, \Sigma(x - \bar{x})^2 = 78,400,000 \).

For sample II, \( n_2 = 1500, \Sigma x = 70,500, \Sigma(x - \bar{x})^2 = 24,000,000 \). Discuss the significance of the difference of the sample means.

**Hypotthesize Testing**

**Sol. Null hypothesis H_0:** There is no significant difference between the sample means.

**H_0:** \( \bar{x}_1 = \bar{x}_2, H_1: \bar{x}_1 \neq \bar{x}_2 \)

To calculate sample variance
\[
s_1^2 = \frac{1}{n_1} \Sigma x_1 - \bar{x}_1^2 = \frac{784000}{1000} = 784
\]

\[
s_2^2 = \frac{1}{n_2} \Sigma x_2 - \bar{x}_2^2 = \frac{15000}{2400000} = 1600
\]

\[
\bar{x}_1 = \frac{\Sigma x_1}{n_1} = 49000 = 49; \bar{x}_2 = \frac{\Sigma x_2}{n_2} = 70500 = 47
\]

Under the null hypothesis, the test statistic
\[
z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_1^2/n_1 + s_2^2/n_2}} = \frac{49 - 47}{\sqrt{784/1000 + 1600/1500}} = 1.470.
\]

**Conclusion.** As the calculated value of \( |z| = 1.47 < 1.96 \), the significant value of \( z \) at 5% level of significance, \( H_0 \) is accepted, i.e., there is no significant difference between the sample means.

**Example 4.** From the data given below, compute the standard error of the difference of the two sample means and find out if the two means significantly differ at 5% level of significance.

<table>
<thead>
<tr>
<th>Group</th>
<th>No. of items</th>
<th>Mean</th>
<th>S.D.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>50</td>
<td>181.5</td>
<td>3.0</td>
</tr>
<tr>
<td>II</td>
<td>75</td>
<td>179</td>
<td>3.6</td>
</tr>
</tbody>
</table>

**Sol. Null hypothesis H_0:** There is no significant difference between the samples.

\( \bar{x}_1 = \bar{x}_2; H_1: \bar{x}_1 \neq \bar{x}_2 \)

**Under H_0:**
\[
z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_1^2/n_1 + s_2^2/n_2}} = \frac{181.5 - 179.0}{\sqrt{\frac{9}{150} + \frac{36}{75}}} = 4.2089.
\]

**Conclusion.** As \( |z| > 1.96 \) the tabulated value of \( z \) at 5% level of significance \( H_0 \) is rejected, i.e., there is significant difference between the samples.

**Example 5.** A random sample of 200 villages from Coimbatore district gives the mean population per village at 485 with a S.D. of 50. Another random sample of the same size from the same district gives the mean population per village at 510 with a S.D. of 49. Is there the difference between the mean values given by the two samples statistically significant? Justify your answer.

**Sol.** Here \( n_1 = 200, n_2 = 250, \bar{x}_1 = 485, \bar{x}_2 = 510, s_1 = 50, s_2 = 49 \)

**Null hypothesis H_0:** There is no significant difference between the mean values (i.e., \( \bar{x}_1 = \bar{x}_2; H_1: \bar{x}_1 \neq \bar{x}_2 \) (Two tailed test))
HYPOTHESIS TESTING

1. No significant difference
2. (b) Highly significant
3. Highly significant
4. Highly significant
5. No significance
6. Not significant

ILLUSTRATIVE EXAMPLES

Example 1. Random samples drawn from two countries gave the following data relating to the heights of adult males:

<table>
<thead>
<tr>
<th>Country A</th>
<th>Country B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean height (in inches)</td>
<td>67.24</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>2.56</td>
</tr>
<tr>
<td>Number of samples</td>
<td>1000</td>
</tr>
</tbody>
</table>

6. Is the difference between the means significant?
7. Is the difference between the standard deviations significant?

Sol. Given: \(n_A = 1000, n_B = 1200, x_A = 67.42, x_B = 67.25, s_A = 2.58, s_B = 2.50\)

Since the samples sizes are large we can take \(s_A = s_B = 2.58\).

(ii) Null Hypothesis: \(H_0: \mu_A = \mu_B\) i.e., sample means do not differ significantly.

Alternative hypothesis: \(H_1: \mu_A \neq \mu_B\) (two tailed test)

\[
z = \frac{x_A - x_B}{\sqrt{\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}}} = -1.56
\]

since \(|z| < 1.96\) we accept the null hypothesis at 5% level of significance.

(iii) We set up the null hypothesis:

\(H_0: \sigma_A = \sigma_B\), i.e., the sample S.D.'s do not differ significantly.

Alternative hypothesis: \(H_1: \sigma_A \neq \sigma_B\) (two tailed)
HYPOTHESIS TESTING

<table>
<thead>
<tr>
<th>Mean yield per plot</th>
<th>Set of 40 plots</th>
<th>Set of 60 plots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1258 lb</td>
<td>1243 lb</td>
<td></td>
</tr>
</tbody>
</table>

2. The yield of wheat in a random sample of 1000 farms in a certain area has a S.D. of 192 kg. Another random sample of 1000 farms gives a S.D. of 224 kg. Are the S.D.s significantly different?

**Answers**
1. $z = 2.321$ Difference significant at 5% level; $z = 1.31$ Difference not significant at 5% level
2. $z = 4.851$ The S.D.s are significantly different.

6.14. TEST OF SIGNIFICANCE OF SMALL SAMPLES

When the size of the sample is less than 30, then the sample is called small sample. For such sample it will not be possible for us to assume that the random sampling distribution of a statistic is approximately normal and the values given by the sample data are sufficiently close to the population values and can be used in their place for the calculation of the standard error of the estimate.

**t-TEST**

6.15. STUDENT'S $t$-DISTRIBUTION


This $t$-distribution is used when sample size is $\leq 30$ and the population standard deviation is unknown.

$t$-statistic as $t = \frac{\bar{x} - \mu}{s(\sqrt{n})}$

d.f. = degrees of freedom where $s = \frac{\sqrt{\sum(X - \bar{X})^2}}{n - 1}$

The $t$-table

The $t$-table given at the end is the probability integral of $t$-distribution. The $t$-distribution has a different values for each degrees of freedom and when the degrees of freedom are infinitely large, the $t$-distribution is equivalent to normal distribution and the probabilities shown in the normal distribution tables are applicable.

Application of $t$-distribution

Some of the applications of $t$-distribution are given below:

1. To test if the sample mean ($\bar{X}$) differs significantly from the hypothetical value $\mu$ of the population mean.
2. To test the significance between two sample means.
3. To test the significance of observed partial and multiple correlation coefficients.

**Critical value of $t$**

The critical value or significant value of $t$ at level of significance $\alpha$ degrees of freedom $\gamma$ for two tailed test is given by

$$P(|t| > t_{\alpha}(\gamma)) = \alpha$$

$$P(|t| < t_{\alpha}(\gamma)) = 1 - \alpha$$

---

**EXERCISE 6.5**

1. The mean yield of two sets of plots and their variability are as given examine.
   (i) Whether the difference in the mean yield of the two sets of plots is significant.
   (ii) Whether the difference in the variability in yields is significant.
The significant value of $t$ at level of significance $\alpha$ for a single tailed test can be got from those of two tailed test by referring to the values at $2\alpha$.

### 6.16. TEST I: $t$-TEST OF SIGNIFICANCE OF THE MEAN OF A RANDOM SAMPLE

To test whether the mean of a sample drawn from a normal population deviates significantly from a stated value when variance of the population is unknown.

$H_0$: There is no significant difference between the sample mean $\bar{x}$ and the population mean $\mu$. i.e., we use the statistic

$$
 t = \frac{\bar{x} - \mu}{s/\sqrt{n}},
$$

where $\bar{x}$ is mean of the sample.

$$
 s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2
$$

with degrees of freedom $(n - 1)$.

At given level of significance $\alpha$, and degrees of freedom $(n - 1)$. We refer to $t$-table $t_\alpha$ (two tailed or one tailed).

If calculated value of $t$ is such that $|t| < t_\alpha$ the null hypothesis is accepted. $|t| > t_\alpha$, $H_0$ is rejected.

### Fiducial limits of population mean

If $t_\alpha$ is the table of $t$ at level of significance $\alpha$ at $(n - 1)$ degrees of freedom.

$$
\left| \frac{\bar{x} - \mu}{s/\sqrt{n}} \right| < t_\alpha
$$

for acceptance of $H_0$.

$$
\bar{x} - \frac{s}{\sqrt{n}} < \mu < \bar{x} + \frac{s}{\sqrt{n}}.
$$

95% confidence limits (level of significance 5%) are

$$
\bar{x} \pm t_{0.05} \frac{s}{\sqrt{n}}.
$$

99% confidence limits (level of significance 1%) are

$$
\bar{x} \pm t_{0.01} \frac{s}{\sqrt{n}}.
$$

Note. Instead of calculating $s$, we calculate $S$ for the sample.

Since

$$
 S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2.
$$

$$
 S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2.
$$

### ILLUSTRATIVE EXAMPLES

#### Example 1. A random sample of size 16 has 53 as mean. The sum of squares of the derivation from mean is 135. Can this sample be regarded as taken from the population having 56 as mean? Obtain 95% and 99% confidence limits of the mean of the population.

**Sol.** $H_0$: There is no significant difference between the sample mean and hypothetical population mean.

$H_0$: $\mu = 56$; $H_1$: $\mu \neq 56$ (Two tailed test)

$$
 t = \frac{\bar{x} - \mu}{s/\sqrt{n}} - t(n - 1 \text{ d.f})
$$

Given: $\bar{X} = 53$, $\mu = 56$, $n = 16$, $\Sigma X - \bar{X}^2 = 135$

$$
 s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n - 1}} = \sqrt{\frac{135}{15}} = 3; t = \frac{53 - 56}{3/\sqrt{16}} = 3 
$$

$|t| = 4; d.f = 16 - 1 = 15$

**Conclusion.** $t_{0.05} = 1.753$.

Since $|t| = 4 > t_{0.05} = 1.753$ i.e., the calculated value of $t$ is more than the table value. The hypothesis is rejected. Hence the sample mean has not come from a population having 56 as mean.

95% confidence limits of the population mean

$$
\bar{X} \pm \frac{s}{\sqrt{n}} t_{0.05}, 53 \pm \frac{3}{\sqrt{16}} (1.725) = 51.706; 54.293
$$

99% confidence limits of the population mean

$$
\bar{X} \pm \frac{s}{\sqrt{n}} t_{0.01}, 53 \pm \frac{3}{\sqrt{16}} (2.602) = 51.048; 54.951.
$$

#### Example 2. The life time of electric bulbs for a random sample of 10 from a large consignment gave the following data:

<table>
<thead>
<tr>
<th>Item</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Life in '000 hrs</td>
<td>4.2</td>
<td>4.6</td>
<td>3.9</td>
<td>4.1</td>
<td>5.2</td>
<td>3.8</td>
<td>3.9</td>
<td>4.3</td>
<td>4.4</td>
<td>5.6</td>
</tr>
</tbody>
</table>

Can we accept the hypothesis that the average life time of bulb is 4000 hrs?

**Sol.** $H_0$: There is no significant difference in the sample mean and population mean.

i.e., $\mu = 4000$ hrs.

Applying the $t$-test:

$$
 t = \frac{\bar{X} - \mu}{s/\sqrt{n}} - t(10 - 1 \text{ d.f})
$$

$$
<table>
<thead>
<tr>
<th>\bar{X}</th>
<th>4.2</th>
<th>4.6</th>
<th>3.9</th>
<th>4.1</th>
<th>5.2</th>
<th>3.8</th>
<th>3.9</th>
<th>4.3</th>
<th>4.4</th>
<th>5.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X - \bar{X}$</td>
<td>-0.2</td>
<td>0.2</td>
<td>-0.5</td>
<td>-0.3</td>
<td>0.8</td>
<td>-0.6</td>
<td>-0.5</td>
<td>0.1</td>
<td>0</td>
<td>1.2</td>
</tr>
<tr>
<td>$(X - \bar{X})^2$</td>
<td>0.04</td>
<td>0.04</td>
<td>0.25</td>
<td>0.09</td>
<td>0.64</td>
<td>0.36</td>
<td>0.05</td>
<td>0.01</td>
<td>0</td>
<td>1.44</td>
</tr>
</tbody>
</table>

$$
 \bar{X} = \frac{\Sigma X}{n} = \frac{44}{10} = 4.4
$$

$$
 \Sigma(X - \bar{X})^2 = 3.12
$$

$$
 s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n-1}} = \sqrt{\frac{3.12}{9}} = 0.589; t = \frac{4.4 - 4}{0.589} = 2.123
$$

For $\gamma = .05$, $t_{0.05} = 2.263$.

**Conclusion.** Since the calculated value of $t$ is less than table $t_{0.05}$, the hypothesis $\mu = 4000$ hrs is accepted.

i.e., The average life time of bulbs could be 4000 hrs.
Example 3. A sample of 20 items has mean 42 units and S.D. 5 units. Test the hypothesis that it is a random sample from a normal population with mean 45 units.

Sol. \( H_0 \): There is no significant difference between the sample mean and the population mean.

\[ \mu = 45 \text{ units} \]
\[ H_1: \mu \neq 45 \text{ (Two tailed test)} \]

Given: \( n = 20, \bar{X} = 42, S = 5; r = 19 \) d.f.

\[ s^2 = \frac{n}{n-1} \bar{S}^2 = \frac{20}{20-1} (5)^2 = 26.31 \therefore s = 5.129 \]

Applying t-test \[ t = \frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{42 - 45}{5.129/4.47} = -2.615; | t | = 2.615 \]

The tabulated value of \( t \) at 5% level for 19 d.f is \( t_{0.05} = 2.09 \).

Conclusion. Since \( | t | \gg t_{0.05} \), the hypothesis \( H_0 \) is rejected, i.e., there is significant difference between the sample mean and population mean.

i.e., The sample could not have come from this population.

Example 4. The 9 items of a sample have the following values 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these values differ significantly from the assumed mean 47.5.

(M.D.U. Dec. 2011)

Sol. \( H_0: \mu = 47.5 \)

i.e., there is no significant difference between the sample and population mean.

\[ H_1: \mu \neq 47.5 \text{ (two tailed test)} \]

Given: \( n = 9, \mu = 47.5 \)

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
X & 45 & 47 & 50 & 52 & 48 & 47 & 49 & 53 & 51 \\
\hline
\bar{X} - \bar{X} & -4.1 & -2.1 & 0.9 & 2.9 & -1.1 & -2.1 & -0.1 & 3.9 & 1.9 \\
(C - \bar{X})^2 & 16.81 & 4.41 & 0.81 & 8.41 & 1.21 & 4.41 & 0.01 & 15.21 & 3.61 \\
\hline
\end{array}
\]

\[ \bar{X} = \frac{\sum X}{n} = \frac{442}{9} = 49.11; \Sigma (X - \bar{X})^2 = 54.88; s^2 = \frac{(\Sigma (X - \bar{X})^2)}{(n-1)} = 6.86 \therefore s = 2.619 \]

Applying t-test \[ t = \frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{49.1 - 47.5}{2.619/4} = 1.7279 \]

\[ t_{0.05} = 2.31 \text{ for } r = 8 \]

Conclusion. Since \( | t | < t_{0.05} \), the hypothesis is accepted i.e., there is no significant difference between their mean.

Example 5. The following results are obtained from a sample of 10 boxes is biscuits.

Mean weight content = 490 gm

S.D. of the weight = 9 gm could the sample come from a population having a mean of 500 gm?

### HYPOTHESIS TESTING

Sol. Given: \( n = 10, \bar{X} = 490; S = 9 \text{ gm, } \mu = 500 \)

\[ s = \sqrt{\frac{\sum S^2}{n-1}} = \sqrt{\frac{10}{9} \times 9^2} = 9.486 \]

\( H_0 \): The difference is not significant i.e., \( \mu = 500 \)

\( H_1: \mu \neq 500 \)

Applying t-test \[ t = \frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{490 - 500}{9.486/\sqrt{10}} = -0.333 \]

\[ t_{0.05} = 2.26 \text{ for } r = 9 \]

Conclusion. Since \( | t | < t_{0.05} \), the hypothesis \( H_0 \) is rejected i.e., \( \mu \neq 500 \)

\[ \therefore \text{ The sample could not have come from the population having mean 500 gm.} \]

### EXERCISE 6.6

1. Find the student's t for the following variable values in a sample of eight:
   - \(-4, -2, -2, 0, 2, 2, 3, 3\) taking the mean of the universe to be zero.


2. Ten individuals are chosen at random from a normal population of students and their marks found to be 63, 65, 66, 67, 68, 69, 70, 71, 71. In the light of these data discuss the suggestion that mean mark of the population of students is 66.

3. The following values gives the lengths of 12 samples of Egyptian cotton taken from a consignment: 48, 49, 49, 49, 52, 45, 47, 47, 46, 45, 50. Test if the mean length of the consignment can be taken as 46.

4. A sample of 18 items has a mean 24 units and standard deviation 3 units. Test the hypothesis that it is a random sample from a normal population with mean 27 units.

5. A random sample of 10 boys had the I.Q's 70, 120, 110, 101, 88, 83, 95, 98, 107 and 100. Do these data support the assumption of a population mean I.Q of 160?

6. A filling machine is expected to fill 5 kg of powder into bags. A sample of 10 bags gave the following weights: 4.7, 4.9, 5.0, 5.1, 5.4, 5.2, 4.6, 5.1, 4.6 and 4.7. Test whether the machine is working properly.

   (M.D.U. Dec. 2010)

7. A machinist is making engine parts with axle-diameter of 0.7 unit. A random sample of 10 parts shows mean diameter 0.742 unit with a standard deviation of 0.04 unit. On the basis of this sample, what would you say that the work is inferior?

   \[
   \begin{align*}
   &1. t = 0.27 \quad 2. \text{ accepted} \quad 3. \text{ accepted} \\
   &4. \text{ rejected} \quad 5. \text{ accepted} \quad 6. \text{ accepted} \\
   &7. \text{ rejected}.
   \end{align*}
   \]

### 6.17. TEST II: t-TEST FOR DIFFERENCE OF MEANS OF TWO SMALL SAMPLES

From a Normal Population

This test is used to test whether the two samples \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \) of sizes \( n_1, n_2 \) have been drawn from two normal populations with mean \( \mu_1 \) and \( \mu_2 \) respectively under the assumption that the population variance are equal \((\sigma_1 = \sigma_2 = \sigma)\).
Hypothesis Testing

Conclusion. Since calculated \( |t| > t_{0.05} \), \( H_0 \) is rejected i.e., \( H_1 \) is accepted.

Type I is definitely superior to type II.

where \( \bar{X} = \frac{\sum X_i}{n_1}, \bar{Y} = \frac{\sum Y_j}{n_2}, \quad s^2 = \frac{1}{n_1 + n_2 - 2} \left[ \sum (X_i - \bar{X})^2 + (Y_j - \bar{Y})^2 \right] \)

is an unbiased estimate of the population variance \( s^2 \).

\( t \) follows \( t \) distribution with \( n_1 + n_2 - 2 \) degrees of freedom.

Example 2. Samples of sizes 10 and 14 were taken from two normal populations with S.D. 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of the two populations are the same at 5% level.

Sol.

\( H_0 \): \( \mu_1 = \mu_2 \) i.e., the means of the two populations are the same.

\( H_1 \): \( \mu_1 \neq \mu_2 \)

Given \( \bar{X}_1 = 20.3, \bar{X}_2 = 18.6; n_1 = 10, n_2 = 14, s_1 = 3.5, s_2 = 5.2 \)

\( s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{10(3.5)^2 + 14(5.2)^2}{10 + 14 - 2} = 22.775 \}

\( s = 4.772 \)

\( t = \frac{\bar{X}_1 - \bar{X}_2}{s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{20.3 - 18.6}{4.772} = 0.8604 \)

The value of \( t \) at 5% level for 22 d.f. is \( t_{0.05} = 2.0739 \).

Conclusion. Since \( |t| = 0.8604 < t_{0.05} \) the hypothesis is accepted, i.e., there is no significant difference between their means.

Example 3. The height of 6 randomly chosen sailors are in inches are 63, 65, 68, 69, 71 and 72. Those of 9 randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Test whether the sailors are on the average taller than soldiers.

Sol. Let \( X \) and \( Y \) be the two samples denoting the heights of sailors and soldiers.

Given the sample size \( n_1 = 6, n_2 = 9, H_0: \mu_1 = \mu_2 \).

i.e., the mean of both the populations are the same.

\( H_1: \mu_1 > \mu_2 \) (one tailed test)

Calculation of two sample mean:

\[
\begin{array}{ccccccc}
X_1 & 63 & 65 & 68 & 69 & 71 & 72 \\
X_2 & 61 & 62 & 65 & 66 & 69 & 70 & 71 & 72 & 73 \\
X_1 - X_2 & -2 & -3 & 0 & 3 & 1 & 4 \\
(X_1 - X_2)^2 & 4 & 9 & 0 & 1 & 9 & 16 \\
\end{array}
\]

\[
\bar{X}_1 = \frac{\sum X_1}{n_1} = 68; \quad \sum (X_1 - \bar{X}_1)^2 = 60
\]

\[
\bar{X}_2 = \frac{\sum X_2}{n_2} = \frac{X_2^2}{n_2}
\]

\[
\begin{array}{ccccccc}
X_2 & 61 & 62 & 65 & 66 & 69 & 70 & 71 & 72 & 73 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
X_1 - X_2 & -6.66 & -5.66 & -2.66 & 1.66 & 1.34 & 2.34 & 3.34 & 4.34 & 5.34 \\
(X_1 - X_2)^2 & 44.36 & 32.035 & 7.0756 & 2.7556 & 1.7956 & 5.4756 & 11.1556 & 18.8356 & 28.5156 \\
\end{array}
\]

ILLUSTRATIVE EXAMPLES

Example 1. Two samples of sodium vapour bulbs were tested for length of life and the following results were got:

<table>
<thead>
<tr>
<th>Type</th>
<th>Size</th>
<th>Sample mean</th>
<th>Sample S.D.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>8</td>
<td>1234 hrs</td>
<td>36 hrs</td>
</tr>
<tr>
<td>II</td>
<td>7</td>
<td>1038 hrs</td>
<td>40 hrs</td>
</tr>
</tbody>
</table>

Is the difference in the means significant to generalise that type I is superior to type II regarding the length of life.

Sol. \( H_0: \mu_1 = \mu_2 \) i.e., two types of bulbs have same lifetime.

\( H_1: \mu_1 > \mu_2 \) i.e., type I is superior to Type II

\[
s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{8 \times (36)^2 + 7 \times (40)^2}{8 + 7 - 2} = 1659.076 \quad s = 40.7317
\]

The t-statistic \( t = \frac{\bar{X}_1 - \bar{X}_2}{s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{1234 - 1036}{40.7317 \sqrt{\frac{1}{8} + \frac{1}{7}}} = 18.1480 - t(n_1 + n_2 - 2) \text{ d.f.)}

\[
t_{0.05} \text{ at d.f. 13 is 1.77 (One tailed test)}
\]
$\bar{X}_2 = \frac{\Sigma X_2}{n_2} = 67.66; \quad \Sigma (X_2 - \bar{X}_2)^2 = 152.0002$

$$s^2 = \frac{1}{n_1 + n_2 - 2} [\Sigma (X_1 - \bar{X}_1)^2 + \Sigma (X_2 - \bar{X}_2)^2]$$

$$= \frac{1}{6 + 9 - 2} [60 + 152.0002] = 16.3077 \Rightarrow s = 4.038$$

Under $H_0$,

$$t = \frac{\bar{X}_1 - \bar{X}_2}{s_{\bar{X}_1 - \bar{X}_2}} = \frac{-68 - 67.66}{4.038} \approx -1.282 \quad \text{with} \quad n_1 + n_2 - 2 = 13 \text{ d.f.}$$

The value of $t$ at 10% level of significance (one tailed) for 13 d.f. is 1.77.

**Conclusion.** Since $|t| = 1.282 < t_{0.05} = 1.77$ the hypothesis $H_0$ is accepted.

**Example 4.** A certain stimulus administered to each of 12 patients resulted in the following increases of blood pressure: 5, 2, 8, -1, 3, 0, -2, 1, 5, 0, 4, 6. Can it be concluded that the stimulus will in general be accompanied by an increase in blood pressure?

*V.T.U. 2007; M.D.U. Dec. 2011*

Sol. To test whether the mean increase in blood pressure of all the patients is greater than zero, we need to assume that this population is normal with mean $\mu$ and S.D. $\sigma$ which are unknown.

$H_0: \mu = 0; \quad H_1: \mu > 0$

The test statistic under $H_0$

$$t = \frac{\bar{X} - \mu_0}{s_{\bar{X}}} \sim t(n - 1 \text{ d.f.})$$

$$\bar{X} = \frac{\Sigma X}{n}$$

$$\sigma^2 = \frac{\Sigma (X - \bar{X})^2}{n - 1}$$

$$= \frac{1}{12} [5^2 + 2^2 + 8^2 + (-1)^2 + 3^2 + 0^2 + 2^2 + 1^2 + 5^2 + 0^2 + 4^2] - \frac{(2.583)^2}{12} = 2.9571$$

$$s = \sqrt{s^2} = 2.9571$$

$$t = \frac{\bar{X} - \mu_0}{s_{\bar{X}}} = \frac{-68 - 67.66}{2.9571} \approx -2.22$$

The tabulated value of $t$ at 11 d.f. is 2.2.

**Conclusion.** The tabulated value of $t_{0.05}$ at 11 d.f. is 2.2.

**Example 5.** Memory capacity of 9 students was tested before and after a course of meditation for a month. State whether the course was effective or not from the data below (in units)

<table>
<thead>
<tr>
<th>Before</th>
<th>10</th>
<th>15</th>
<th>9</th>
<th>7</th>
<th>12</th>
<th>16</th>
<th>17</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>After</td>
<td>12</td>
<td>17</td>
<td>8</td>
<td>5</td>
<td>6</td>
<td>11</td>
<td>18</td>
<td>20</td>
</tr>
</tbody>
</table>

**Example 6.** The following figures refer to observations in live independent samples.

**Sample I**

| 25 | 30 | 28 | 34 | 24 | 20 | 13 | 32 | 22 | 38 |

**Sample II**

| 40 | 34 | 22 | 20 | 31 | 40 | 30 | 23 | 36 | 17 |

**Sol.** $H_0$: Two samples have been drawn from the population of equal means, i.e., there is no significant difference between their means

**Hypothesis Testing**

$$H_0: \mu_1 = \mu_2$$

$$H_1: \mu_1 \neq \mu_2$$

**Example 6.** The following figures refer to observations in live independent samples.

**Sample I**

| 25 | 30 | 28 | 34 | 24 | 20 | 13 | 32 | 22 | 38 |

**Sample II**

| 40 | 34 | 22 | 20 | 31 | 40 | 30 | 23 | 36 | 17 |

**Sol.** $H_0$: The two samples have been drawn from the population of equal means, i.e., there is no significant difference between their means.

**Hypothesis Testing**

$$H_0: \mu_1 = \mu_2$$

$$H_1: \mu_1 \neq \mu_2$$

**Example 6.** The following figures refer to observations in live independent samples.

**Sample I**

| 25 | 30 | 28 | 34 | 24 | 20 | 13 | 32 | 22 | 38 |

**Sample II**

| 40 | 34 | 22 | 20 | 31 | 40 | 30 | 23 | 36 | 17 |
EXERCISE 6.7

1. The mean life of 10 electric motors was found to be 1450 hrs with S.D. of 423 hrs. A second sample of 17 motors chosen from a different batch showed a mean life of 1280 hrs with a S.D. of 398 hrs. Is there a significant difference between the means of the two samples?

2. The marks obtained by a group of 9 regular course students and another group of 11 part-time course students in a test are given below:

<table>
<thead>
<tr>
<th>Regular</th>
<th>56</th>
<th>62</th>
<th>63</th>
<th>54</th>
<th>60</th>
<th>51</th>
<th>67</th>
<th>69</th>
<th>58</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part-time</td>
<td>62</td>
<td>70</td>
<td>71</td>
<td>62</td>
<td>60</td>
<td>56</td>
<td>75</td>
<td>64</td>
<td>68</td>
</tr>
</tbody>
</table>

Examine whether the marks obtained by regular students and part-time students differ significantly at 5% and 1% level of significance.

HYPOTHESIS TESTING

3. A group of 10 rats fed on diet A and another group of 8 rats fed on a different diet B recorded the following increase in weight (gm).

<table>
<thead>
<tr>
<th>Diet A</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>1</th>
<th>12</th>
<th>4</th>
<th>3</th>
<th>9</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diet B</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

Does it show the superiority of diet A over the diet B?

4. Two independent samples of sizes 7 and 9 have the following values:

<table>
<thead>
<tr>
<th>Sample A</th>
<th>10</th>
<th>12</th>
<th>13</th>
<th>13</th>
<th>12</th>
<th>10</th>
<th>14</th>
<th>11</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample B</td>
<td>10</td>
<td>12</td>
<td>15</td>
<td>12</td>
<td>10</td>
<td>14</td>
<td>11</td>
<td>12</td>
<td>11</td>
</tr>
</tbody>
</table>

Test whether the difference between the means is significant.

5. To compare the prices of a certain product in two cities, 10 shops when related at random in each town. The price was noted below:

<table>
<thead>
<tr>
<th>City 1</th>
<th>61</th>
<th>63</th>
<th>56</th>
<th>63</th>
<th>56</th>
<th>63</th>
<th>59</th>
<th>56</th>
<th>44</th>
<th>61</th>
</tr>
</thead>
<tbody>
<tr>
<td>City 2</td>
<td>55</td>
<td>54</td>
<td>47</td>
<td>59</td>
<td>51</td>
<td>61</td>
<td>57</td>
<td>54</td>
<td>64</td>
<td>58</td>
</tr>
</tbody>
</table>

Test whether the average prices can be said to be the same in two cities.

6. The average number of articles produced by two machines per day are 200 and 250 with standard deviations 20 and 25 respectively on the basis of records of 25 days production. Can you regard both the machines equally efficient at 5% level of significance?

7. Two salesmen represent a firm in a certain company. One of them claims that he makes larger sales than the other. A sample survey was made and the following results were obtained:

<table>
<thead>
<tr>
<th>No. of shops</th>
<th>1st Salesman (18)</th>
<th>2nd Salesman (20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average sales</td>
<td>₹ 210</td>
<td>₹ 175</td>
</tr>
<tr>
<td>S.D.</td>
<td>₹ 25</td>
<td>₹ 20</td>
</tr>
</tbody>
</table>

Find if the average sales differ significantly.

8. Two types of drugs A and B were used on 5 and 7 patients respectively for reducing their weights. The decrease in the weight after using drugs for six months are as follows:

<table>
<thead>
<tr>
<th>Drug A</th>
<th>10</th>
<th>12</th>
<th>13</th>
<th>11</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drug B</td>
<td>8</td>
<td>9</td>
<td>12</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

Is there a significant difference in the efficiency of the two drugs? If not, which drug you should buy?

9. Nine patients, to whom a certain drug was administered, registered the following rise in blood pressure:

| 3, 7, 4, 1, 3, 6, 4, 1, 5 |

Test the hypothesis that the drug does not raise the blood pressure.

10. The change in sleeping hours of 7 patients after taking a medicine are as follows:

| 0.7, 0.1, 0.3, 1.2, 1.0, 0.3, 0.4 hrs |

Do these data give evidence that the medicine produces additional hours of sleep?
HYPOTHESIS TESTING

6.16. CHI-SQUARE ($\chi^2$) TEST

When a coin is tossed 200 times, the theoretical considerations lead us to expect 100 heads and 100 tails. But in practice, these results are rarely achieved. The quantity $\chi^2$, known as chi-square, describes the magnitude of discrepancy between theory and observation. If $\chi^2 = 0$, the observed and expected frequencies coincide. The greater the discrepancy between the observed and expected frequencies, the greater is the value of $\chi^2$. Thus $\chi^2$ affords a measure of the correspondence between theory and observation.

If $O_i$ (i = 1, 2, ..., n) is a set of observed (experimental) frequencies and $E_i$ (i = 1, 2, ..., n) is the corresponding set of expected (theoretical or hypothetical) frequencies, then $\chi^2$ is defined as

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

where $\sum O_i = \sum E_i = N$ (total frequency) and degrees of freedom (d.f.) = $n - 1$.

Note. (i) If $\chi^2 = 0$, the observed and theoretical frequencies agree exactly.

(ii) If $\chi^2 > 0$ they do not agree exactly.

6.19. DEGREE OF FREEDOM

While comparing the calculated value of $\chi^2$ with the table value, we have to determine the degrees of freedom.

If we have to choose any four numbers whose sum is 50, we can exercise our independent choice for any three numbers only, the fourth being 50 minus the total of the three numbers selected. Thus, though we have to choose any four numbers, our choice was reduced three because of one condition imposed. There was only one restraint on our freedom and as degrees of freedom were 4 - 1 = 3. If two restrictions are imposed, our freedom to choose will be further curtailed and degrees of freedom will be 4 - 2 = 2.

In general, the number of degrees of freedom is the total number of observations minus the number of independent constraints imposed on the observations. Degrees of freedom (d.f.) are usually denoted by $v$ (the letter ‘nu’ of the Greek alphabet).

Thus, $v = n - k$, where $k$ is the number of independent constraints in a set of data of observations.

Note 1. For a $p \times q$ contingency table (p columns and q rows),

$$v = (p - 1)(q - 1)$$

2. In the case of a contingency table, the expected frequency of any class

$$= \frac{\text{Total of row in which it occurs} \times \text{Total of col. in which it occurs}}{\text{Total number of observations}}$$

6.20. CONDITIONS FOR APPLYING $\chi^2$ TEST

Following are the conditions which should be satisfied before $\chi^2$ test can be applied.

(a) $N$, the total number of frequencies should be large. It is difficult to say what constitutes largeness, but as an arbitrary figure, we may say that $N$ should be at least 50, however, few the cells.

(b) No theoretical cell-frequency should be small. Here again, it is difficult to say what constitutes smallness, but 5 should be regarded as the very minimum and 10 is better. If small theoretical frequencies occur ($i.e., c < 10$), the difficulty is overcome by grouping two or more classes together before calculating $(O - E)$. It is important to remember that the number of degrees of freedom is determined with the number of classes after regrouping.

(c) The constraints on the cell frequencies, if any, should be linear.

Note. If any one of the theoretical frequency is less than 5, then we apply a corrected given by

$F$ Yates, which is usually known as ‘Yates correction for continuity’, we add 0.5 to the cell frequency which is less than 5 and adjust the remaining cell frequency so that the marginal total is not changed.

6.21. THE $\chi^2$ DISTRIBUTION

For large sample sizes, the sampling distribution of $\chi^2$ can be closely approximated by a continuous curve known as the chi-square distribution. The probability function of $\chi^2$ distribution is given by

$$f(\chi^2) = \frac{c(\chi^2)^{v/2 - 1}}{2^{v/2 - 1} \Gamma(v/2)} e^{-\chi^2/2}$$

where $c = 2.71828$, $v = \text{number of degrees of freedom}$; $c$ is a constant depending only on $v$.

Symbolically, the degrees of freedom are denoted by the symbol $v$ or by d.f. and are obtained by the rule $v = n - k$, where $k$ refers to the number of independent constraints.

In general, when we fit a binomial distribution the number of degrees of freedom is one less than the number of classes; when we fit a Poisson distribution the degrees of freedom are 2 less than the number of classes, because we use the total frequency and the arithmetic mean to get the parameter of the Poisson distribution. When we fit a normal curve the number of degrees of freedom is 3 less than the number of classes, because in this fitting we use the total frequency, mean and standard deviation.

If the data is given in a series of "n" number then degrees of freedom $v = n - 1$.

In the case of Binomial distribution d.f. $v = n - 1$

In the case of Poisson distribution d.f. $v = n - 2$

In the case of Normal distribution d.f. $v = n - 3$. 
HYPOTHESIS TESTING

\[ \chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} \]

Conclusion. Tabulated value of \( \chi^2 \) at 5% level of significance for (6 - 1 = 5) d.f. is 11.09.
Since the calculated value of \( \chi^2 = 21.30 > 11.09 \) the tabulated value, \( H_0 \) is rejected.

\[ \text{i.e., Die is not unbiased or Die is biased.} \]

Example 3. The following table shows the distribution of digits in numbers chosen at random from a telephone directory.

<table>
<thead>
<tr>
<th>Digits</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>1026</td>
<td>1107</td>
<td>997</td>
<td>966</td>
<td>1075</td>
<td>933</td>
<td>1107</td>
<td>972</td>
<td>964</td>
<td>853</td>
</tr>
</tbody>
</table>

Test whether the digits may be taken to occur equally frequently in the directory.

Sol. Null hypothesis \( H_0 \): The digits taken in the directory occur equally frequently.
\( \text{i.e., there is no significant difference between the observed and expected frequency.} \)

Under \( H_0 \), the expected frequency is given by \( \frac{10000}{10} = 1000 \).

To find the value of \( \chi^2 \)

\[ \chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{58542}{1000} = 58.542. \]

Conclusion. The tabulated value of \( \chi^2 \) at 5% level of significance for 9 d.f. is 16.919.
Since the calculated value of \( \chi^2 \) is greater than the tabulated value, \( H_0 \) is rejected.
\( \text{i.e., there is significant difference between the observed and theoretical frequency.} \)

Example 4. Records taken of the number of male and female births in 800 families having four children are as follows:

<table>
<thead>
<tr>
<th>No. of male births</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of families</td>
<td>32</td>
<td>178</td>
<td>290</td>
<td>236</td>
<td>94</td>
</tr>
</tbody>
</table>

Test whether the data are consistent with the hypothesis that the binomial law holds and the chance of male birth is equal to that of female birth, namely \( p = q = 1/2 \).

\( \text{P.T.U. 2006; U.P.T.U. 2009} \)

Sol. \( H_0 \): The data are consistent with the hypothesis of equal probability for male and female births, \( \text{i.e., } p = q = 1/2 \).
We use Binomial distribution to calculate theoretical frequency given by:

\[ N(r) = N \times \text{P}(X = r) \]

where \( N \) is the total frequency, \( N(r) \) is the number of families with \( r \) male children:

\[ \text{P}(X = r) = \binom{n}{r} p^r q^{n-r} \]

where \( p \) and \( q \) are probability of male and female birth, \( n \) is the number of children.

\[
\begin{align*}
N(0) &= \text{No. of families with 0 male children} = 800 \times 4C_0 \left(\frac{1}{2}\right)^4 = 800 \times 1 \times \frac{1}{2^4} = 50 \\
N(1) &= 800 \times 4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 200; N(2) = 800 \times 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 300 \\
N(3) &= 800 \times 4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 = 200; N(4) = 800 \times 4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0 = 50
\end{align*}
\]

<table>
<thead>
<tr>
<th>Observed frequency ( O_i )</th>
<th>32</th>
<th>178</th>
<th>290</th>
<th>236</th>
<th>94</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp-frequency ( E_i )</td>
<td>50</td>
<td>200</td>
<td>300</td>
<td>200</td>
<td>50</td>
</tr>
<tr>
<td>( (O_i - E_i)^2 ) ( E_i )</td>
<td>324</td>
<td>454</td>
<td>100</td>
<td>1206</td>
<td>1966</td>
</tr>
<tr>
<td>( (O_i - E_i)^2 ) ( E_i )</td>
<td>6.48</td>
<td>2.42</td>
<td>0.333</td>
<td>6.48</td>
<td>38.72</td>
</tr>
</tbody>
</table>

\[ \chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 54.433. \]

Conclusion. The calculated value of \( \chi^2 \) at 5% level of significance for \( 5 - 1 = 4 \) d.f. is 9.49. Since the calculated value of \( \chi^2 \) is greater than the tabulated value, \( H_0 \) is rejected.

Note. Since the fitting is binomial, the degrees of freedom \( v = n - 1 \) i.e., \( v = 5 - 1 = 4 \).

Example 5. Verify whether Poisson distribution can be assumed from the data given below:

<table>
<thead>
<tr>
<th>No. of defects</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>6</td>
<td>13</td>
<td>13</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

Sol. \( H_0 \): Poisson fit is a good fit to the data. (P.T.U. 2007)

Mean of the given distribution = \( \text{E}(X_i) = \frac{E_i \times O_i}{N} \frac{94}{97} = 2 \)

To fit a Poisson distribution we require \( m \). Parameter \( m = \bar{X} = 2 \).

By Poisson distribution the frequency of \( r \) success is

\[ N(r) = N \times e^{-m} \frac{m^r}{r!}, N \text{ is the total frequency.} \]

\[ \chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 4.7266. \]

Conclusion. The calculated value of \( \chi^2 \) at 5% level of significance for 3 d.f. is 7.815. Since the calculated value of \( \chi^2 \) is less than that of the tabulated value. Hence \( H_0 \) is accepted, i.e., the experimental result support the theory.
EXERCISE 6.8

1. The following table gives the frequency of occupation of the digits 0, 1, ..., 9 in the last place in four logarithms of numbers 10-99. Examine if there is any peculiarity.

<table>
<thead>
<tr>
<th>Digits</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>6</td>
<td>16</td>
<td>15</td>
<td>10</td>
<td>12</td>
<td>12</td>
<td>3</td>
<td>2</td>
<td>9</td>
<td>5</td>
</tr>
</tbody>
</table>

2. The sales in a supermarket during a week are given below. Test the hypothesis that the sales do not depend on the day of the week, using a significant level of 0.05.

<table>
<thead>
<tr>
<th>Days</th>
<th>Mon</th>
<th>Tues</th>
<th>Wed</th>
<th>Thus</th>
<th>Fri</th>
<th>Sat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales (in 1000 ₹)</td>
<td>65</td>
<td>54</td>
<td>60</td>
<td>56</td>
<td>71</td>
<td>84</td>
</tr>
</tbody>
</table>

3. A survey of 320 families with 5 children each revealed the following information:

<table>
<thead>
<tr>
<th>No. of boys</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of girls</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Is this result consistent with the hypothesis that male and female birth are equally probable?

4. 4 coins were tossed at a time and this operation is repeated 160 times. It is found that 4 heads occur 6 times, 3 heads occur 43 times, 2 heads occur 69 times, one head occur 34 times. Discuss whether the coin may be regarded as unbiased?

5. Fit a Poisson distribution to the following data and test the goodness of fit.

<table>
<thead>
<tr>
<th>x</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>109</td>
</tr>
<tr>
<td>1</td>
<td>65</td>
</tr>
<tr>
<td>2</td>
<td>22</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

(M.D.U. Dec. 2010)

6. Fit a Poisson distribution to the following data and test the goodness of fit.

<table>
<thead>
<tr>
<th>x</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>46</td>
</tr>
<tr>
<td>1</td>
<td>38</td>
</tr>
<tr>
<td>2</td>
<td>22</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

(M.D.U. Dec. 2010)

7. (a) Fit a binomial distribution to the following data and test the goodness of fit.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>13</td>
<td>25</td>
<td>52</td>
<td>58</td>
<td>32</td>
<td>16</td>
<td>4</td>
</tr>
</tbody>
</table>

(b) A set of five similar coins is tossed 320 times and the result is

<table>
<thead>
<tr>
<th>No. of heads</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>6</td>
<td>27</td>
<td>72</td>
<td>112</td>
<td>71</td>
<td>32</td>
</tr>
</tbody>
</table>

Test the hypothesis that the data follow a Binomial distribution. (M.D.U. Dec. 2010)

8. 200 digits are chosen at random from a set of tables. The frequencies of the digits are as follows:

<table>
<thead>
<tr>
<th>Digit</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>18</td>
<td>19</td>
<td>23</td>
<td>21</td>
<td>16</td>
<td>25</td>
<td>22</td>
<td>20</td>
<td>21</td>
<td>15</td>
</tr>
</tbody>
</table>

Use chi-square test to assess the correctness of the hypothesis that the digits were distributed in equal numbers in the tables from where they are chosen.

9. In the accounting department of bank, 100 accounts are selected at random and estimated for errors. The following results were obtained:

<table>
<thead>
<tr>
<th>No. of errors</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of accounts</td>
<td>35</td>
<td>40</td>
<td>19</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Does this information verify that the errors are distributed according to the Poisson probability law?

10. A sample analysis of examination results of 500 students, it was found that 280 students have failed. 170 have secured a third class, 80 have secured a second class and the rest a first class. Do these figures support the general belief that above categories are in the ratio 4 : 3 : 2 : 1 respectively?

Answers

1. no
2. accepted
3. accepted
4. unbiased
5. Poisson law fits the data
6. Poisson law fits the data
7. (a) Binomial law does not fit the data (b) rejected
8. accepted
9. may be
10. yes.

6.23. $\chi^2$-TEST AS A TEST OF INDEPENDENCE

With the help of $\chi^2$ test, we can find whether or not two attributes are associated. We take the null hypothesis that there is no association between the attributes under study, i.e., we assume that the two attributes are independent. If the calculated value of $\chi^2$ is less than the table value at a specified level of significance, the hypothesis holds good, i.e., the attributes are independent and do not bear any association. On the other hand, if the calculated value of $\chi^2$ is greater than the table value at a specified level of significance, we say that the results of the experiment do not support the hypothesis. In other words, the attributes are associated. Thus a very useful application of $\chi^2$ test is to investigate the relationship between trials or attributes which can be classified into two or more categories.

The sample data set out into two-way table, called contingency table.

Let us consider two attributes A and B divided into r classes $A_1$, $A_2$, $A_3$, ..., $A_r$ and B divided into s classes $B_1$, $B_2$, $B_3$, ..., $B_s$. If $(A_i, B_j)$ represents the number of persons possessing the attributes $A_i$ and $B_j$ respectively, $(r = 1, 2, ..., r; s = 1, 2, ..., s$) and $(A, B)$ represent the number of persons possessing attributes A and B. Also we have $\sum_{i=1}^{r} A_i = \sum_{j=1}^{s} B_j = N$ where N is the total frequency. The contingency table for $r \times s$ is given below:
Hypothesis testing

$H_0$: Both the attributes are independent, i.e., $A$ and $B$ are independent under the null hypothesis, we calculate the expected frequency as follows:

$$P(A_i) = \text{Probability that a person possesses the attribute } A_i = \frac{(A_i)}{N}, \quad i = 1, 2, \ldots, r$$

$$P(B_j) = \text{Probability that a person possesses the attribute } B_j = \frac{(B_j)}{N}$$

$$P(A_i, B_j) = \text{Probability that a person possesses both attributes } A_i \text{ and } B_j = \frac{(A_i, B_j)}{N}$$

If $(A_i, B_j)_0$ is the expected number of persons possessing both the attributes $A_i$ and $B_j$:

$$(A_i, B_j)_0 = NP(A_i, B_j) = NP(A_i)(B_j) = \frac{N(A_i)(B_j)}{N} = \frac{(A_i, B_j)}{N}$$

Hence,

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{[(A_i, B_j) - (A_i, B_j)_0]^2}{(A_i, B_j)_0}$$

which is distributed as a $\chi^2$ variate with $(r - 1)(s - 1)$ degrees of freedom.

Note 1. For a $2 \times 2$ contingency table where the frequencies are $\frac{a}{b} \frac{c}{d}$ $\chi^2$ can be calculated from independent frequencies as $\chi^2 = \frac{(a + b + c + d)(ad - bc)^2}{(a + b)(c + d)(a + d)(b + c)}$.

Note 2. If the contingency table is not $2 \times 2$, then the formula for calculating $\chi^2$ as given in note 1, can't be used. Hence we have another formula for calculating the expected frequency $E(a_i, b_j) = \frac{(A_i)(B_j)}{N}$, i.e., expected frequency in each cell is $\frac{\text{Product of column total and row total}}{\text{whole total}}$.

Note 3. If $\frac{a}{c} \frac{b}{d}$ is the $2 \times 2$ contingency table with two attributes, $Q = \frac{ad - bc}{ad + bc}$ is called the coefficient of association.

If the attributes are independent then $\frac{a}{b} = \frac{c}{d}$.

Note 4. Yates's Correction. In a $2 \times 2$ table, if the frequencies of a cell is small, we make Yates's correction to make $\chi^2$ continuous.

Decrease by $\frac{1}{2}$ those cell frequencies which are greater than expected frequencies, and increase by $\frac{1}{2}$ those which are less than expectation. This will not affect the marginal columns. This correction is known as Yates's correction to continuity.

After Yates's correction

$$\chi^2 = \frac{N[(bc - ad - \frac{1}{2}N)]}{(a + c)(b + d)(c + d)(a + b)}$$

when $ad - bc > 0$.

$$\chi^2 = \frac{N[(ad - bc - \frac{1}{2}N)]}{(a + c)(b + d)(c + d)(a + b)}$$

when $ad - bc < 0$.

Illustrative Examples

Example 1. What are the expected frequencies of $2 \times 2$ contingency tables given below:

<table>
<thead>
<tr>
<th></th>
<th>(i)</th>
<th>(ii)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a$</td>
<td>$b$</td>
</tr>
<tr>
<td>$c$</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>$d$</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Sol.

<table>
<thead>
<tr>
<th></th>
<th>(i)</th>
<th></th>
<th>(ii)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a$</td>
<td>$b$</td>
<td>$a + b$</td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$d$</td>
<td>$c + d$</td>
<td></td>
</tr>
<tr>
<td>$a + c$</td>
<td>$b + d$</td>
<td>$a + b + c + d$</td>
<td>$N$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Expected frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(a + c)(b + d)(a + d)$</td>
</tr>
<tr>
<td></td>
<td>$(b + d)(c + d)$</td>
</tr>
<tr>
<td></td>
<td>$a + b + c + d$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>(i)</th>
<th></th>
<th>(ii)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2$</td>
<td>10</td>
<td>$12$</td>
<td></td>
</tr>
<tr>
<td>$6$</td>
<td>6</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$8$</td>
<td>16</td>
<td>24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Expected frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$8 \times 12$</td>
</tr>
<tr>
<td></td>
<td>$24$</td>
</tr>
<tr>
<td></td>
<td>$16 \times 12$</td>
</tr>
<tr>
<td></td>
<td>$24$</td>
</tr>
<tr>
<td></td>
<td>$8$</td>
</tr>
<tr>
<td></td>
<td>$24$</td>
</tr>
<tr>
<td></td>
<td>$8$</td>
</tr>
</tbody>
</table>
Example 2. From the following table regarding the colour of eyes of fathers and sons, test if the colour of son's eye is associated with that of the father.

<table>
<thead>
<tr>
<th>Eye colour of son</th>
<th>Light</th>
<th>Not light</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eye colour of father</td>
<td>Light</td>
<td>471</td>
</tr>
<tr>
<td>Not light</td>
<td>148</td>
<td>230</td>
</tr>
</tbody>
</table>

Sol. Null hypothesis \( H_0 \): The colour of son's eye is not associated with that of the father, i.e., they are independent.

Under \( H_0 \), we calculate the expected frequency in each cell as:

\[
\text{Expected frequencies} = \frac{\text{Product of column total and row total}}{\text{whole total}}
\]

<table>
<thead>
<tr>
<th>Eye colour of father</th>
<th>Light</th>
<th>Not light</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light</td>
<td>619 x 522 = 339.02</td>
<td>259 x 522 = 167.62</td>
<td>522</td>
</tr>
<tr>
<td>Not light</td>
<td>619 x 378 = 259.98</td>
<td>259 x 378 = 121.38</td>
<td>378</td>
</tr>
<tr>
<td>Total</td>
<td>619</td>
<td>289</td>
<td>900</td>
</tr>
</tbody>
</table>

\[
\chi^2 = \frac{(471 - 339.02)^2}{339.02} + \frac{(51 - 167.62)^2}{167.62} + \frac{(148 - 259.98)^2}{259.98} + \frac{(230 - 121.38)^2}{121.38} = 261.489
\]

Conclusion. The tabulated value of \( \chi^2 \) at 5% level for 1 d.f. is 3.841.

Since the calculated value of \( \chi^2 \) > tabulated value of \( \chi^2 \), \( H_0 \) is rejected. They are dependent, i.e., the colour of son's eye is associated with that of the father.

Example 3. The following table gives the number of good and bad parts produced by each of the three shifts in a factors

<table>
<thead>
<tr>
<th>Goods parts</th>
<th>Bad parts</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day shift</td>
<td>960</td>
<td>40</td>
</tr>
<tr>
<td>Evening shift</td>
<td>940</td>
<td>50</td>
</tr>
<tr>
<td>Night shift</td>
<td>950</td>
<td>45</td>
</tr>
<tr>
<td>Total</td>
<td>2850</td>
<td>135</td>
</tr>
</tbody>
</table>

Test whether or not the production of bad parts is independent of the shift on which they were produced.

Sol. Null hypothesis \( H_0 \): The production of bad parts is independent of the shift on which they were produced.

i.e., the two attributes, production and shifts are independent.

Example 4. From the following data, find whether hair colour and sex are associated.

<table>
<thead>
<tr>
<th>Sex</th>
<th>Colour</th>
<th>Fair</th>
<th>Red</th>
<th>Medium</th>
<th>Dark</th>
<th>Black</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boys</td>
<td></td>
<td>592</td>
<td>849</td>
<td>504</td>
<td>119</td>
<td>36</td>
<td>2100</td>
</tr>
<tr>
<td>Girls</td>
<td></td>
<td>544</td>
<td>677</td>
<td>451</td>
<td>97</td>
<td>14</td>
<td>1783</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>1136</td>
<td>1526</td>
<td>955</td>
<td>216</td>
<td>50</td>
<td>3883</td>
</tr>
</tbody>
</table>
Null hypothesis $H_0$: The two attributes hair colour and sex are not associated.

Let A and B be the attributes hair colour and sex respectively. A is divided into 5 classes ($r = 5$). B is divided into 2 classes ($s = 2$).

Degrees of freedom = $(r - 1)(s - 1) = (5 - 1)(2 - 1) = 4$

Under $H_0$, we calculate $\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$

To calculate the expected frequency $(A_iB_j)_0$ as follows:

\[
\begin{align*}
(A_1B_1)_0 &= \frac{(A_1)(B_1)}{N} = \frac{1136 \times 2100}{3883} = 614.37; \\
(A_1B_2)_0 &= \frac{(A_1)(B_2)}{N} = \frac{1136 \times 1783}{3883} = 521.629; \\
(A_2B_1)_0 &= \frac{(A_2)(B_1)}{N} = \frac{1526 \times 2100}{3883} = 852.289; \\
(A_2B_2)_0 &= \frac{(A_2)(B_2)}{N} = \frac{1526 \times 1783}{3883} = 700.71; \\
(A_3B_1)_0 &= \frac{(A_3)(B_1)}{N} = \frac{955 \times 2100}{3883} = 516.482; \\
(A_3B_2)_0 &= \frac{(A_3)(B_2)}{N} = \frac{955 \times 1783}{3883} = 493.517; \\
(A_4B_1)_0 &= \frac{(A_4)(B_1)}{N} = \frac{216 \times 2100}{3883} = 116.816; \\
(A_4B_2)_0 &= \frac{(A_4)(B_2)}{N} = \frac{216 \times 1783}{3883} = 99.183; \\
(A_5B_1)_0 &= \frac{(A_5)(B_1)}{N} = \frac{50 \times 2100}{3883} = 27.04; \\
(A_5B_2)_0 &= \frac{(A_5)(B_2)}{N} = \frac{50 \times 1783}{3883} = 22.959.
\end{align*}
\]

Null hypothesis $H_0$: The two attributes are independent; i.e., vaccination can’t be regarded as preventive measure of smallpox.

Degrees of freedom $v = (r - 1)(s - 1) = (2 - 1)(2 - 1) = 1$

Under $H_0$, $\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$

Calculation of expected frequency

\[
\begin{align*}
(A_1B_1)_0 &= \frac{(A_1)(B_1)}{N} = \frac{343 \times 368}{1482} = 85.1713; \\
(A_1B_2)_0 &= \frac{(A_1)(B_2)}{N} = \frac{343 \times 1114}{1482} = 257.828; \\
(A_2B_1)_0 &= \frac{(A_2)(B_1)}{N} = \frac{1139 \times 368}{1482} = 282.828; \\
(A_2B_2)_0 &= \frac{(A_2)(B_2)}{N} = \frac{1139 \times 1114}{1482} = 856.171.
\end{align*}
\]

Calculation of $\chi^2$

\[
\begin{align*}
\chi^2 &= 9.7999.
\end{align*}
\]

Conclusion. Table of $\chi^2$ at 5% level of significance for 4 d.f. is 9.488. Since the calculated value of $\chi^2$ < tabulated value $H_0$ is rejected, i.e., the two attributes are not independent. i.e., the hair colour and sex are associated.

Example 5. Can vaccination be regarded as preventive measure of smallpox as evidenced by the following data of 1482 persons exposed to smallpox in a locality. 368 in all were attacked of these 1482 persons and 343 were vaccinated and of these only 35 were attacked.

Sol. For the given data, we form the contingency table. Let the two attributes be vaccination and exposed to smallpox. Each attributes is divided into two classes.

<table>
<thead>
<tr>
<th>Disease small per B</th>
<th>Vaccination A</th>
<th>Vaccinated</th>
<th>Not</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attacked</td>
<td>35</td>
<td>333</td>
<td>368</td>
<td></td>
</tr>
<tr>
<td>Not</td>
<td>308</td>
<td>806</td>
<td>1114</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>343</td>
<td>1139</td>
<td>1482</td>
<td></td>
</tr>
</tbody>
</table>

Calculated value of $\chi^2 = 48.2261$. 

\[
\chi^2 = 9.7999.
\]
Conclusion. Tabulated value of $\chi^2$ at 5% level of significance for 1 d.f. is 3.841.
Since the calculated value of $\chi^2 >$ tabulated value $H_0$ is rejected.
*i.e.,* the two attributes are not independent. *i.e.,* the vaccination can be regarded as preventive measure of smallpox.

**EXERCISE 6.9**

1. In a locality 100 persons were randomly selected and asked about their educational achievements. The results are given below:

<table>
<thead>
<tr>
<th>Sex</th>
<th>Education</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Middle</td>
</tr>
<tr>
<td>Male</td>
<td>10</td>
</tr>
<tr>
<td>Female</td>
<td>25</td>
</tr>
</tbody>
</table>

Based on this information can you say the education depends on sex.

2. The following data is collected on two characters:

<table>
<thead>
<tr>
<th></th>
<th>Smokers</th>
<th>Non-smokers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Literate</td>
<td>85</td>
<td>57</td>
</tr>
<tr>
<td>Iiterate</td>
<td>45</td>
<td>66</td>
</tr>
</tbody>
</table>

Based on this information can you say that there is no relation between habit of smoking and literacy.

3. 500 students at school were graded according to their intelligences and economic conditions of their homes. Examine whether there is any association between economic condition and intelligence, from the following data:

<table>
<thead>
<tr>
<th>Economic conditions</th>
<th>Intelligence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Good</td>
</tr>
<tr>
<td>Rich</td>
<td>85</td>
</tr>
<tr>
<td>Poor</td>
<td>165</td>
</tr>
</tbody>
</table>

4. In an experiment on the immunisation of goats from anthrax, the following results were obtained. Derive your inferences on the efficiency of the vaccine.

<table>
<thead>
<tr>
<th>Inoculated with vaccine</th>
<th>Died</th>
<th>Survived</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>Not inoculated</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

**Answers**


**CHAPTER 7**

**Linear Programming**

**7.1. INTRODUCTION**

Linear programming (briefly written as LP) came into existence during World War II (1939-45) when the British and American military management called upon a group of scientists to study and plan the war activities so that maximum damages could be inflicted on the enemy camps at minimum cost and loss with the limited resources available with them. Because of the success in military operations, the subject was found extremely useful for allocation of scarce resources for optimal results. Business and industry, agriculture and military sectors have, however, made the most significant use of the technique. But now it is being extensively used in all functional areas of management, airlines, oil refining, education, pollution control, transportation planning, health care system etc. The utility of the technique is enhanced by the availability of highly efficient computer codes. A lot of research work is being carried out all over the world. Kantorovich and Koopmans were awarded the noble prize in the year 1975 in economics for their pioneering work in linear programming. In India, it came into existence in 1949, with the opening of an operations research unit at the Regional Research Laboratory at Hyderabad.

**7.2. DEFINITIONS AND BASIC CONCEPTS**

The word ‘programming’ means planning and refers to a process of determining a particular program or strategy or course of action among various alternatives to achieve the desired objective. The word ‘linear’ means that all relationships involved in a particular programme are linear.

Linear Programming is the technique of optimizing (i.e., maximizing or minimizing) a linear function of several variables subject to a number of constraints stated in the form of linear inequalities/equations.

**Linear Programming Problem.** Any problem in which we apply linear programming is called a linear programming problem (briefly written as LPP).

The mathematical model of a general linear programming problem with $n$ variables and $m$ constraints can be stated as

Optimize (maximize or minimize)

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$
linear programming

1. Identify the decision variables and denote them by \( x_1, x_2, x_3, \ldots \)
2. Identify the objective function and express it as a linear function of the decision variables. In its general form, it is represented as:
   \[
   \text{Optimize (Maximize or Minimize)} \quad Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n
   \]
   (the optimal value of \( Z \) is obtained by the graphical method or simplex method. The graphical method is more suitable when there are two variables)
3. Identify the set of constraints and express them as linear equations/inequalities in terms of decision variables.
4. Add the non-negativity restrictions on the decision variables, as in the physical problems, negative values of decision variables have no valid interpretation.
   Thus \( x_1 \geq 0, \ x_2 \geq 0, \ldots, x_n \geq 0 \).

ILLUSTRATIVE EXAMPLES

Example 1. A dealer wishes to purchase a number of fans and sewing machines. He has only \( \text{\text{}]} \) 9750 to invest and has space for at most 30 items. A fan costs him \( \text{\text{}} \) 480 and a sewing machine \( \text{\text{}]} \) 380. His expectation is that he can sell a fan at a profit of \( \text{\text{}]} \) 35 and a sewing machine at a profit of \( \text{\text{}} \) 24. Assume that he can sell all the items that he buys. Formulate this problem as an LPP, so that he can maximize his profit.

Sol. Let \( x \) and \( y \) denote the number of fans and sewing machines respectively. (\( x \) and \( y \) are the decision variables)

\[
\begin{align*}
\text{Cost of } x \text{ fans} & = \text{\text{}} 480x \\
\text{Cost of } y \text{ sewing machines} & = \text{\text{}} 380y \\
\Rightarrow \text{The total cost of } x \text{ fans and } y \text{ sewing machines} & = (480x + 380y) \\
\text{Since the dealer has only } \text{\text{}} 9750 \text{ to invest, the total cost cannot exceed } \text{\text{}} 9750. \\
\therefore 480x + 380y & \leq 9750 \\
\text{Dividing by } 30, \text{ we have } 16x + 12y & \leq 325 \\
\text{Since the dealer has space for at most 30 items} \\
\therefore x + y & \leq 30 \\
\text{Since the number of fans and the number of sewing machines cannot be negative} \\
\therefore x \geq 0, \ y \geq 0 \quad \text{(non-negativity constraints)} \\
\text{Profit on } x \text{ fans} & = \text{\text{}} 35x \\
\text{Profit on } y \text{ sewing machines} & = \text{\text{}} 24y \\
\Rightarrow \text{The total profit on } x \text{ fans and } y \text{ sewing machines} & = (35x + 24y) \\
\text{The dealer wishes to maximize his profit} \\
\therefore Z = 35x + 24y \quad \text{(objective function)} \\
\text{Maximize } Z = 35x + 24y \\
\text{subject to the constraints} \\
16x + 12y & \leq 325 \\
x + y & \leq 30 \quad x \geq 0, \ y \geq 0.
\end{align*}
\]

7.3. FORMULATION OF A LINEAR PROGRAMMING PROBLEM

Formulation of an LPP as a mathematical model is the first and the most important step in the solution of the LPP. It is an art in itself and needs sufficient practice. However, in general, the following steps are involved:
Example 2. A manufacturer of patent medicines is preparing a production plan on medicines A and B. There are sufficient raw materials available to make 20000 bottles of A and 40000 bottles of B, but there are only 45000 bottles into which either of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of A, it takes 1 hour to prepare enough material to fill 1000 bottles of B and there are 66 hours available for this operation. The profit is ₹8 per bottle for A and ₹7 per bottle for B. The manufacturer wants to schedule his production in order to maximize his profit. Formulate this problem as an LP model.

Sol. Let \( x \) and \( y \) denote the number of bottles of type A and type B medicines respectively.

Total profit (in ₹) is \( Z = 8x + 7y \)

Raw material constraints are

\[
x \leq 20000, \quad y \leq 40000
\]

Since only 45000 bottles are available

\[
x + y \leq 45000
\]

It takes 3 hours to prepare enough material to fill 1000 bottles of type A.

\[
3x = \frac{3 \times 1000}{1000}
\]

The number of hours required to prepare enough material to fill \( x \) bottles of type A

Similarly, the number of hours required to prepare enough material to fill \( y \) bottles of type B

\[
3y = \frac{3 \times 1000}{1000}
\]

Since total number of hours available for this operation is 66

\[
\frac{3x}{1000} + \frac{y}{1000} \leq 66 \quad \text{or} \quad 3x + y \leq 66000
\]

Obviously

\[
x \geq 0, \quad y \geq 0
\]

The LP model of the problem is

Maximize

\[
Z = 8x + 7y
\]

subject to the constraints

\[
x \leq 20000
\]

\[
y \leq 40000
\]

\[
x + y \leq 45000
\]

\[
3x + y \leq 66000
\]

\[
x \geq 0, \quad y \geq 0.
\]

Example 3. An electronic company produces three types of parts for automatic washing machines. It purchases casting of the parts from a local foundry and then finishes the part on drilling, shaping and polishing machines.

The selling prices of parts A, B and C respectively are ₹8, ₹10 and ₹14. All parts made can be sold. Castings for parts A, B and C respectively cost ₹5, ₹6 and ₹8. The company possesses only one of each type of machine. Costs per hour to run each of the three machines are ₹50 for drilling, ₹30 for shaping and ₹30 for polishing. The capacities (parts per hour) for each part on each machine are shown in the following table:

<table>
<thead>
<tr>
<th>Machine</th>
<th>Part A</th>
<th>Part B</th>
<th>Part C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drilling</td>
<td>25</td>
<td>40</td>
<td>25</td>
</tr>
<tr>
<td>Shaping</td>
<td>25</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Polishing</td>
<td>40</td>
<td>30</td>
<td>40</td>
</tr>
</tbody>
</table>

The management of the shop wants to know how many parts of each type it should produce per hour in order to maximize profit for an hour's run. Formulate this problem as an LP model so as to maximize total profit to the company.

Sol. Let \( x_1, x_2 \) and \( x_3 \) denote respectively the numbers of type A, B and C parts to be produced per hour.

Consider one type A part

Selling price = ₹8

Cost of casting = ₹5

Since 25 type A parts per hour can be run on the drilling machine at a cost of ₹20

\[
\text{Drilling cost per type A part} = \frac{20}{25} = ₹0.80
\]

Similarly shaping cost

\[
= \frac{30}{20} = ₹1.50
\]

Polishing cost

\[
= \frac{20}{40} = ₹0.50
\]

\[
\text{Profit per type A part} = ₹(8 - (5 + 0.80 + 1.50 + 0.50)) = ₹0.25
\]

By similar reasoning

Profit per type B part

\[
= ₹[10 - (6 + 20 \times \frac{20}{40} + \frac{30}{20} + 30)] = ₹1.00
\]

Profit per type C part

\[
= ₹[14 - (10 \times \frac{20}{25} + \frac{30}{20} + \frac{30}{40})] = ₹0.95
\]

Total profit is given by \( Z = 0.25x_1 + 1.00x_2 + 0.95x_3 \).

On the drilling machine, one type A part consumes \( \frac{1}{25} \) th of the available hour, one type B part consumes \( \frac{1}{40} \) th and one type C part consumes \( \frac{1}{25} \) th of the available hour.

The drilling machine constraint is

\[
\frac{x_1}{25} + \frac{x_2}{40} + \frac{x_3}{25} \leq 1
\]

Similarly, the shaping machine constraint is

\[
\frac{x_1}{25} + \frac{x_2}{20} + \frac{x_3}{20} \leq 1
\]

and the polishing machine constraint is
The production constraint is:
\[ x_1 + x_2 + x_3 \leq 1 \]
\[ \frac{40}{40} + \frac{36}{40} + \frac{40}{40} \]
Also, the non-negativity constraint is:
\[ x_1, x_2, x_3 \geq 0 \]

The LP model of the given problem is:
Maximize
\[ Z = 0.25x_1 + 1.00x_2 + 0.95x_3 \]
subject to the constraints
\[ \begin{align*}
x_1 + x_2 + x_3 & \leq 1 \\
x_1 & \leq 25 \\
x_2 & \leq 40 \\
x_3 & \leq 25 \\
x_1 + x_2 & \leq 40 \\
x_2 + x_3 & \leq 30 \\
x_1, x_2, x_3 & \geq 0.
\end{align*} \]

**Example 4.** Consider the following problem faced by a production planner in a soft drink plant. He has two bottling machines A and B. A is designed for 8-ounce bottles and B for 16-ounce bottles. However, each can also be used for both types of bottles with some loss of efficiency. The manufacturing data is as follows:

<table>
<thead>
<tr>
<th>Machine</th>
<th>8-ounce Bottles</th>
<th>16-ounce Bottles</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>100/minute</td>
<td>40/minute</td>
</tr>
<tr>
<td>B</td>
<td>60/minute</td>
<td>75/minute</td>
</tr>
</tbody>
</table>

The machines can be run for 8 hours per day, 5 days per week. Profit on an 8-ounce bottle is 15 paisa and on a 16-ounce bottle 25 paisa. Weekly production of the drink cannot exceed 3,00,000 bottles and the market can absorb 25,000, 8-ounce bottles and 7,000, 16-ounce bottles per week. The planner wishes to maximize his profit, subject of course, to all the production and marketing restrictions. Formulate this problem as an LP model to maximize total profit.

**Solution.** Let \( x_1 \) and \( x_2 \) denote the number of 8-ounce and 16-ounce bottles respectively to be produced weekly.

Total profit (in ₹) is given by
\[ Z = 0.15x_1 + 0.25x_2 \]

**Machine time constraints**

Total available time in a week on machine A = \( 8 \times 5 \times 60 \) minutes = 2400 minutes

Time for 100, 8-ounce bottles is 1 minute

\[ \therefore \text{Time for } x_1 \text{ 8-ounce bottles is } \frac{x_1}{100} \text{ minutes} \]

Similarly, time for \( x_2 \) 16-ounce bottles is \( \frac{x_2}{40} \) minutes

\[ \Rightarrow \text{Machine time constraint on machine A is } \frac{x_1}{100} + \frac{x_2}{40} \leq 2400 \]

Similarly, machine time constraint on machine B is
\[ \frac{x_1}{60} + \frac{x_2}{75} \leq 2400 \]

**Marketing constraints.** The market can absorb 25,000, 8-ounce bottles and 7,000, 16-ounce bottles.

\[ x_1 \leq 25,000 \text{ and } x_2 \leq 7,000 \]

**Non-negativity constraints.** The number of bottles cannot be negative
\[ x_1 \geq 0, \quad x_2 \geq 0 \]

Hence the LP model of the given problem is:
Maximize
\[ Z = 0.15x_1 + 0.25x_2 \]
subject to the constraints
\[ \begin{align*}
\frac{x_1}{100} + \frac{x_2}{40} & \leq 2400 \\
\frac{x_1}{60} + \frac{x_2}{75} & \leq 2400 \\
x_1 + x_2 & \leq 3,00,000 \\
x_1 \leq 25,000, \quad x_2 \leq 7000 \\
x_1 \geq 0, \quad x_2 \geq 0.
\end{align*} \]

**Example 5.** A firm making castings uses electric furnace to melt iron with the following specifications:

<table>
<thead>
<tr>
<th>Specifications</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbon</td>
<td>3.20%</td>
<td>3.40%</td>
</tr>
<tr>
<td>Silicon</td>
<td>2.25%</td>
<td>2.35%</td>
</tr>
</tbody>
</table>

Specifications and costs of various raw materials used for this purpose are given below:

<table>
<thead>
<tr>
<th>Material</th>
<th>Carbon %</th>
<th>Silicon %</th>
<th>Cost (₹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel scrap</td>
<td>0.4</td>
<td>0.15</td>
<td>850/tonne</td>
</tr>
<tr>
<td>Cast iron scrap</td>
<td>3.80</td>
<td>2.40</td>
<td>900/tonne</td>
</tr>
<tr>
<td>Remelt from foundry</td>
<td>3.50</td>
<td>2.30</td>
<td>500/tonne</td>
</tr>
</tbody>
</table>

If the total charge of iron metal required is 4 tonnes, find the weight in kg of each raw material that must be used in the optimal mix at minimum cost. Formulate this problem as an LP model.

**Solution.** Let \( x_1, x_2, x_3 \) be the weights (in kg) of the raw materials:

Steel scrap, cast iron scrap and remelt from foundry respectively.

Cost of 1 tonne i.e., 1000 kg steel scrap is ₹ 850

\[ \Rightarrow \text{cost of } x_1 \text{ kg steel scrap is } \frac{850}{1000} x_1 = 0.85x_1 \]

Similarly, cost of \( x_2 \) kg of cast iron scrap is
\[ \frac{900}{1000} x_2 = 0.9x_2 \]

and cost of \( x_3 \) kg of remelt from foundry is
\[ \frac{500}{1000} x_3 = 0.5x_3 \]
Total cost of raw material is given by
\[ Z = 0.85x_1 + 0.9x_2 + 0.5x_3 \]
and the objective is to minimize it.

Total iron metal required is 4 tonnes i.e., 4000 kg
\[ x_1 + x_2 + x_3 = 4000 \]
The iron melt is to have a minimum of 3.2% carbon
\[ 0.4x_1 + 3.8x_2 + 3.5x_3 \geq 3.2 \times 4000 \text{ i.e., } 12800 \]
The iron melt is to have a maximum of 3.4% carbon
\[ 0.4x_1 + 3.8x_2 + 3.5x_3 \leq 3.4 \times 4000 \text{ i.e., } 13600 \]
The iron melt is to have a minimum of 2.25% silicon
\[ 0.15x_1 + 2.4x_2 + 2.3x_3 \geq 2.25 \times 4000 \text{ i.e., } 9000 \]
The iron melt is to have a maximum of 2.35% silicon
\[ 0.15x_1 + 2.4x_2 + 2.3x_3 \leq 2.35 \times 4000 \text{ i.e., } 9400 \]
Since the amounts of raw material cannot be negative
\[ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0 \]

Hence the LP model of the given problem is:
Minimize
\[ Z = 0.85x_1 + 0.9x_2 + 0.5x_3 \]
subject to the constraints
\[ x_1 + x_2 + x_3 = 4000 \]
\[ 0.4x_1 + 3.8x_2 + 3.5x_3 \geq 12800 \]
\[ 0.4x_1 + 3.8x_2 + 3.5x_3 \leq 13600 \]
\[ 0.15x_1 + 2.4x_2 + 2.3x_3 \geq 9000 \]
\[ 0.15x_1 + 2.4x_2 + 2.3x_3 \leq 9400 \]
\[ x_1, x_2, x_3 \geq 0. \]

Example 6. There is a factory located at each of the two places P and Q. From these locations, a certain commodity is delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are, respectively, 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are, respectively, 8 and 6 units. The cost of transportation per unit is given below:

<table>
<thead>
<tr>
<th>From</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>16</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>Q</td>
<td>10</td>
<td>12</td>
<td>10</td>
</tr>
</tbody>
</table>

How many units should be transported from each factory to each depot in order that the transportation cost is minimum? Formulate the above linear programming problem mathematically.

Sol. Let \( x \) and \( y \) units of the commodity be transported from the factory P to the depots A and B respectively. Since the factory P has the capacity of producing 8 units of commodity, therefore, \( (8 - x - y) \) units will be transported from the factory P to the depot C.

Clearly \( x \geq 0, \ y \geq 0 \) and \( 8 - x - y \geq 0 \text{ i.e., } x + y \leq 8. \)

Now, the weekly requirement of the depot A is 5 units of the commodity and \( x \) units are already transported from the factory P, therefore, the remaining \( (5 - x) \) units are to be transported from the factory Q. Similarly \( (5 - y) \) and \( 6 - (5 - x + 5 - y) = x + y - 4 \) units are to be transported from the factory Q to the depots B and C respectively.

Clearly \( 5 - x \geq 0, \ 5 - y \geq 0 \text{ and } x + y - 4 \geq 0 \text{ i.e., } x \leq 5, \ y \leq 5 \text{ and } x + y \geq 4. \)

The total transportation cost is given by
\[ Z = 16x + 10y + 15(8 - x - y) + 10(5 - x) + 12(5 - y) + 10(x + y - 4) \]
or
\[ Z = x - 7y + 190. \]

The objective is to minimize \( Z = x - 7y \) since 190 is a constant and does not affect the optimal solution.

Hence the above LPP can be stated mathematically as follows:
Minimize \( Z = x - 7y \)
subject to the constraints
\[ x + y \leq 8 \]
\[ x \leq 5 \]
\[ y \leq 5 \]
\[ x + y \geq 4 \]
\[ x \geq 0, \ y \geq 0. \]
### Exercise 7.1

1. A company produces two types of models, M₁ and M₂. Each M₁ model requires 4 hours of grinding and 2 hours of polishing, whereas each M₂ model requires 3 hours of grinding and 5 hours of polishing. The company has 2 grinders and 3 polishers. Each grinder works for 40 hours a week, polishing. The company has 5 grinders and 4 polishers. Each polisher works for 60 hours a week. Profit on an M₁ model is ₹ 3, and on an M₂ model is ₹ 4. Whatever is produced in a week is sold in the market. How should the company allocate its resources so as to maximize the profit? Formulate the problem as an LP model.

2. A housewife wishes to mix two types of foods X and Y in such a way that the vitamin content of the mixture contains at least 6 units of vitamin A and 11 units of vitamin B. Food X costs ₹ 60 per kg, and food Y costs ₹ 80 per kg. Food X contains 3 units per kg of vitamin A and 5 units per kg of vitamin B, while food Y contains 4 units per kg of vitamin A and 2 units per kg of vitamin B. Food X and food Y are available in unlimited quantities. Formulate the above problem as an LP model to minimize the cost of mixture (M.D.U. Dec. 2006).

3. A manufacturer has 3 machines installed in its factory. Machines I and II are capable of being used for producing the items A and B, respectively. Each machine produces items A and B. He operates for at least 12 hours a day, whereas machine III must be operated for at least 5 hours a day. He requires the use of the three machines. The number of hours required for producing one unit of each of the items on the three machines is given in the following table:

<table>
<thead>
<tr>
<th>Item</th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

He makes a profit of ₹ 6 on item A and ₹ 4 on item B. Assuming that he can sell all he produces, how many of each item should he produce so as to maximize his profit? Formulate this LP model mathematically.

4. An airplane can carry a maximum of 200 passengers. A profit of ₹ 400 is made on each first class ticket, and a profit of ₹ 300 on each economy class ticket. The airline reserves at least 20 seats for first class. However, at least 4 times as many passengers prefer to travel by economy class than by the first class. How many tickets of each class must be sold in order to maximize profit for the airline? Formulate the problem as an LP model (M.D.U. Dec. 2005).

5. Sudha wants to invest ₹ 10000 in Saving Certificates and in National Saving Bonds. According to rules, she has to invest at least ₹ 1000 in Saving Certificates and at least ₹ 2000 in National Saving Bonds. If the rate of interest on Saving Certificates is 8% and the rate of interest on the National Saving Bonds is 10% p.a., how should she invest her money to earn maximum yearly income? Formulate the problem as an LP model.

6. A firm manufactures 3 products A, B, and C. The profits are ₹ 1, ₹ 2, and ₹ 3, respectively. The firm has two machines, M₁ and M₂, and below is the required processing time in minutes for each machine on each product.

<table>
<thead>
<tr>
<th>Machine</th>
<th>Product</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>M₁</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>M₂</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Machines M₁ and M₂ have 2000 and 2500 machine-minutes respectively. The firm must manufacture 100 A's, 200 B's, and 50 C's but not more than 150 A's. Set up an LP model to maximize profit.

---

### Linear Programming

7. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grams of aspirin, 5 grams of bicarbonate, and 1 gram of codeine. Size B contains 1 gram of aspirin, 8 grams of bicarbonate, and 6 grams of codeine. It is found by users that it requires at least 12 grams of aspirin, 74 grams of bicarbonate, and 24 grams of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the above LPP mathematically.

8. The manager of an oil refinery must decide on the optimal mix of two possible blending processes of the inputs and outputs per production run as follows:

<table>
<thead>
<tr>
<th>Processes</th>
<th>Inputs (units)</th>
<th>Outputs (units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crude 1</td>
<td>Crude 2</td>
<td>Petrol (superior)</td>
</tr>
<tr>
<td>A</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>12</td>
<td>15</td>
</tr>
</tbody>
</table>

The availability of the various grades of crude is limited to the extent of 400 units and 450 units respectively per day. The market demand indicates that at least 260 units and 240 units of the superior and ordinary quality petrol is required every day. The profit contribution indicates that process A contributes ₹ 480 per run while the process B contributes ₹ 400 per run. The manager is interested in determining an optimal product mix for maximizing the company's profit. Formulate the problem as an LP model.

9. A tape recorder company manufactures models A, B, and C which have profit contributions per unit of ₹ 15, ₹ 40, and ₹ 60 respectively. The weekly minimum production requirements are 25 units, 130 units, and 55 units for models A, B, and C respectively. Each type of recorder requires a certain amount of time for the manufacturing of component parts, as well as assembling and packing. Specifically, A, B, and C require 4, 3, and 1 hour for manufacturing, 3, 2, and 1 hour for assembling, and 2, 1, and 1 hour for packing. The corresponding figures for the planned units of models A, B, and C are 25, 4, and 2. The company has 130 hours of manufacturing, 170 hours of assembling, and 120 hours of packing time. Formulate this problem as an LP model so as to maximize total profit to the company.

10. ABC Foods Company is developing a low-calorie high-protein diet supplement called Hi-Pro. The specifications for Hi-Pro have been established by a panel of medical experts. These specifications along with the calorie, protein, and vitamin content of three basic foods, are given in the following table:

<table>
<thead>
<tr>
<th>Nutritional Elements</th>
<th>Units of Nutritional Elements (Per 100 gm Serving of Basic Foods)</th>
<th>Basic Foods Hi-Pro Specifications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calories</td>
<td>250, 300, 200</td>
<td>300</td>
</tr>
<tr>
<td>Protein</td>
<td>150, 150, 150</td>
<td>100</td>
</tr>
<tr>
<td>Vitamin A</td>
<td>75, 75, 150</td>
<td>100</td>
</tr>
<tr>
<td>Vitamin C</td>
<td>125, 125, 150</td>
<td>100</td>
</tr>
<tr>
<td>Cost per serving (₹)</td>
<td>1.50, 2.50, 2.00</td>
<td>1.20</td>
</tr>
</tbody>
</table>

The following must be satisfied:

- Calorie content 300 units
- Protein content 100 units
- Vitamin A content 100 units
- Vitamin C content 100 units
- Cost per serving ≤ 1.20

What quantities of foods 1, 2, and 3 should be used? Formulate this problem as an LP model to minimize cost of serving.

11. A firm manufactures two items A and B. It purchases castings which are then machined, bored and polished. Castings for items A and B cost ₹ 3 and ₹ 4 each and are sold at ₹ 6 and ₹ 7 respectively. Running costs of these machines are ₹ 20, ₹ 14, and ₹ 17.50 per hour respectively. Running costs of these machines are ₹ 20, ₹ 14, and ₹ 17.50 per hour respectively.
Formulate the problem so that the product mix maximizes the profit. The capacities of the machines are:

<table>
<thead>
<tr>
<th>Item</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Machining</td>
<td>25 per hr.</td>
<td>40 per hr.</td>
</tr>
<tr>
<td>Boring</td>
<td>20 per hr.</td>
<td>35 per hr.</td>
</tr>
<tr>
<td>Polishing</td>
<td>35 per hr.</td>
<td>25 per hr.</td>
</tr>
</tbody>
</table>


12. A brick manufacturer has two depots, A and B, with stocks of 30000 and 20000 bricks respectively. He receives orders from three building's P, Q and R for 15000, 20000 and 15000 bricks respectively. The cost of transporting 1000 bricks to the builders from the depots (in ₹) are given below:

<table>
<thead>
<tr>
<th>From</th>
<th>P</th>
<th>Q</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>20</td>
<td>60</td>
<td>40</td>
</tr>
<tr>
<td>B</td>
<td>40</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

How should the manufacturer fulfil the orders so as to keep the cost of transportation minimum? Formulate the above LPP mathematically.

**Answers**

1. Maximize $Z = 3x + 4y$
   subject to the constraints
   
   $2x + y \leq 40$
   $2x + 5y \leq 180$
   $x \geq 0$, $y \geq 0$.

2. Minimize $Z = 50x + 60y$
   subject to the constraints
   
   $3x + 4y \geq 8$
   $5x + 2y \geq 11$
   $x \geq 0$, $y \geq 0$.

3. Maximize $Z = 6x + 4y$
   subject to the constraints
   
   $x + 2y \leq 12$
   $2x + y \leq 12$
   $4x + 5y \geq 20$
   $x \geq 0$, $y \geq 0$.

4. Maximize $Z = 400x + 300y$
   subject to the constraints
   
   $x + 3y \leq 200$
   $x \geq 0$, $y \geq 0$.

5. Maximize $Z = \frac{8}{100} x + \frac{10}{100} y$
   subject to the constraints
   
   $x + y \leq 12000$
   $x \geq 1000$, $y \geq 2000$
   $x \geq 0$, $y \geq 0$.

6. Maximize $Z = 3x_1 + 2x_2 + 4x_3$
   subject to the constraints
   
   $4x_1 + 3x_2 + 6x_3 \leq 2000$
   $2x_1 + 2x_2 + 4x_3 \leq 2500$
   $100 \leq x_1 \leq 150$
   $x_2 \geq 200$, $x_3 \geq 60$.

7. Minimize $Z = x + y$
   subject to the constraints
   
   $2x + y \geq 12$
   $5x + 8y \leq 74$
   $x + 6y \geq 24$
   $x \geq 0$, $y \geq 0$.

8. Maximize $Z = 480x + 400y$
   subject to the constraints
   
   $10x + 12y \leq 400$
   $6x + 15y \leq 450$
   $10x + 12y \geq 200$
   $16x + 12y \geq 240$
   $x \geq 0$, $y \geq 0$.

9. Maximize $Z = 15x_1 + 40x_2 + 60x_3$
   subject to the constraints
   
   $x_1 \geq 25$
   $x_2 \geq 130$
   $x_3 \geq 55$
   $4x_1 + 2.5x_2 + 6x_3 \leq 1560$
   $3x_1 + 4x_2 + 9x_3 \leq 2040$
   $x_1 + 2x_2 + 4x_3 \leq 624$.

10. Minimize $Z = 1.5x_1 + 2x_2 + 1.2x_3$
    subject to the constraints
    
    $350x_1 + 250x_2 + 200x_3 \geq 300$
    $250x_1 + 300x_2 + 150x_3 \geq 200$
    $100x_1 + 150x_2 + 75x_3 \geq 100$
    $75x_1 + 125x_2 + 150x_3 \geq 100$
    $x_1, x_2, x_3 \geq 0$.

11. Maximize $Z = 1.2x + 1.4y$
    subject to the constraints
    
    $40x + 25y \leq 1000$
    $35x + 25y \leq 780$
    $25x + 35y \leq 875$
    $x, y \geq 0$.

12. Minimize $Z = 60x - 20y$
    subject to the constraints
    
    $x + y \geq 15$
    $x \leq 15$
    $y \leq 20$
    $x + y \leq 30$
    $x \geq 0$, $y \geq 0$.

where 1 unit of bricks = 1000 bricks

Total cost of transportation is $Z = Z + 1500$.

**7.4. GRAPH OF A LINEAR INEQUALITY**

The constraints in the mathematical model of an LPP are in the form of linear inequalities. Let us see how to graph linear inequalities involving two variables.
ILLUSTRATIVE EXAMPLES

Example 1. Graph the linear inequality:

\[ 3x + 4y \leq 12. \]

**Sol.** The given inequality is \(3x + 4y \leq 12\).

Replacing \(y \leq -\frac{3}{4}x + 3\), the corresponding equation is \(3x + 4y = 12\).

Putting \(y = 0\) in (2), \(3x = 12\) or \(x = 4\).

Putting \(x = 0\) in (2), \(4y = 12\) or \(y = 3\).

The graph of equation (2) passes through the points \((4, 0)\) and \((0, 3)\). The line through these two points in the graph of equation (2). This line divides the plane into two half-planes.

Putting \(x = 0\) and \(y = 0\) in (1), we get \(0 \leq 12\) which is true. Therefore the origin \((0, 0)\) lies in the feasible region. Hence, the shaded region below the line \(AB\) and the points on the line \(AB\) (as shown in the adjoining figure) constitute the graph of the inequality (1).

Example 2. Solve graphically the following system of linear inequations:

\[ x \geq 4, y \geq 2. \]

**Sol.** The given system of inequations is

\[ x \geq 4, \quad y \geq 2. \]

The equation corresponding to inequation (1) is \(x = 4\). This is a line parallel to \(y\)-axis and at a distance 4 units to the right of \(y\)-axis.

The equation corresponding to inequation (2) is \(y = 2\). This is a line parallel to \(x\)-axis and at a distance 2 units above it.

The inequation (1) represents the half-plane above this line, including this line.

The inequation (2) represents the half-plane below the line \(y = 2\), excluding this line.

Hence the common region is the shaded region. Any point in this shaded region represents a solution of the given system of inequations.

Example 3. Solve graphically \(x + 3y \leq 12, 3x + y \leq 12, x \geq 0, y \geq 0\).

**Sol.** The given system of inequations is

\[ x + 3y \leq 12 \]
\[ 3x + y \leq 12 \]
\[ x \geq 0, y \geq 0 \]

and

The equation corresponding to inequation (1) is \(x + 3y = 12\).

Putting \(x = 0, y = 3\) or \(y = 4\).

Putting \(y = 0, x = 12\).

The graph of \(x + 3y = 12\) is the straight line through the points \((0, 4)\) and \((12, 0)\).

Putting \(x = 0\) and \(y = 0\) in (1); \(0 \leq 12\) which is true.

The graph of inequation (1) contains the origin.

The inequation (1) represents the half-plane on the origin side of this line, including the line.

The equation corresponding to inequation (2) is \(3x + y = 12\).

Putting \(x = 0, y = 12\).

Putting \(y = 0, 3x = 12\) or \(x = 4\).

The graph of \(3x + y = 12\) is the straight line through the points \((0, 12)\) and \((4, 0)\).

Putting \(x = 0\) and \(y = 0\) in (2); \(0 \leq 12\) which is also true.

The graph of inequation (2) also contains the origin.

The inequation (2) represents the half-plane on the origin side of this line, including the line.

The inequation (3) is \(x \geq 0, y \geq 0\) \(\Rightarrow\) feasible region is in first quadrant only.

The common feasible region (i.e., intersection of all feasible regions) in first quadrant is shown as shaded in the figure. Every point in this shaded region represents a solution of the given system of linear inequations.
Example 4. Solve graphically: \(x + y \geq 3, 7x + 6y \leq 42, x \leq 5, y \leq 4, x \geq 0, y \geq 0\).

Sol. The graph of \(x + y = 3\) is the line AB through the points \((3, 0)\) and \((0, 3)\). Since \((0, 0)\) does not satisfy \(x + y \geq 3\), the region represented by this inequation is the half-plane above the line AB, including this line.

The graph of \(7x + 6y = 42\) is the line CD through the points \((6, 0)\) and \((0, 7)\). (By putting \(x = 0\) and \(y = 0\) respectively in \(7x + 6y = 42\). Since \((0, 0)\) satisfies \(7x + 6y \leq 42\), the region represented by this inequation is the half-plane on the origin side of line CD, including this line.

\(x \leq 5\) represents the half-plane on the left of the line \(x = 5\), including this line.

\(y \leq 4\) represents the half-plane below the line \(y = 4\), including this line.

\(x \geq 0\) and \(y \geq 0\) = Feasible region is in first quadrant only.

The common feasible region (i.e., intersection of all feasible regions) is shown shaded in the figure. Every point in this shaded region represents a solution of the given system of linear inequations.

Example 5. Solve graphically the inequations \(3x + 2y \leq 24, x + 2y \leq 16, x + y \leq 10, x \geq 0, y \geq 0\).

Sol. The given system of inequations is

\[
\begin{align*}
3x + 2y & \leq 24 \quad \ldots (1) \\
x + 2y & \leq 16 \quad \ldots (2) \\
x + y & \leq 10 \quad \ldots (3) \\
x \geq 0, y \geq 0 & \quad \ldots (4)
\end{align*}
\]

The equation corresponding to inequation (1) is \(3x + 2y = 24\).
Putting \(x = 0, 2y = 24\) or \(y = 12\)
Putting \(y = 0, 3x = 24\) or \(x = 8\).

\[
\begin{align*}
\therefore & \text{ Graph of } 3x + 2y = 24 \text{ is the straight line through the points } (0, 12) \text{ and } (8, 0). \\
\text{Putting } x = 0 \text{ and } y = 0 \text{ in } (1), 0 \leq 24 \text{ which is true.} \\
\therefore & \text{ Graph of inequation (1) contains the origin.} \\
\therefore & \text{ The inequation (1) represents the half-plane on the origin side of this line, including the line.}
\end{align*}
\]

The equation corresponding to inequation (2) is \(x + 2y = 16\).
Putting \(x = 0, 2y = 16\) or \(y = 8\)
Putting \(y = 0, x = 16\)

\[
\begin{align*}
\therefore & \text{ Graph of } x + 2y = 16 \text{ is the straight line through the points } (0, 8) \text{ and } (16, 0). \\
\text{Putting } x = 0 \text{ and } y = 0 \text{ in } (2), 0 \leq 16 \text{ which is true.} \\
\therefore & \text{ Graph of inequation (2) also contains the origin.} \\
\therefore & \text{ The inequation (2) represents the half-plane on the origin side of this line, including the line.}
\end{align*}
\]

The equation corresponding to inequation (3) is \(x + y = 10\).
Putting \(x = 0, y = 10\)
Putting \(y = 0, x = 10\)

\[
\begin{align*}
\therefore & \text{ Graph of } x + y = 10 \text{ is the straight line through the points } (0, 10) \text{ and } (10, 0). \\
\text{Putting } x = 0 \text{ and } y = 0 \text{ in } (3), 0 \leq 10 \text{ which is also true.} \\
\therefore & \text{ Graph of inequation (3) also contains the origin.} \\
\therefore & \text{ The inequation (3) represents the half-plane on the origin side of this line, including the line.}
\end{align*}
\]

Inequation (4) is \(x \geq 0, y \geq 0\) = Feasible region is in first quadrant only.

\[
\begin{align*}
\therefore & \text{ The common feasible region (i.e., intersection of all feasible regions) in first quadrant is shown as shaded in the figure. Every point in this shaded region represents a solution of the given system of linear inequations.}
\end{align*}
\]
EXERCISE 7.2

Graph the following systems of inequations and shade the feasible region:

1. \( x \geq 1, x \leq 4, y \geq 1, y \leq 3 \) (or \( 1 \leq x \leq 4, 1 \leq y \leq 3 \)).
2. \( 0 \leq y \leq 3, y \leq x, 2x + y \leq 9 \).
3. \( 3x + 2y \leq 18, x + 2y \leq 10, x \geq 0, y \geq 0 \).
4. \( x + y \leq 9, y \leq x, x \geq 1 \).
5. \( 3x + 4y \leq 60, x + 2y \leq 30, x \geq 0, y \geq 0 \).
6. \( 2x + y \leq 24, x + y \leq 11, 2x + 6y \leq 40, x \geq 0, y \geq 0 \).

Answers

7.5. THE GRAPHICAL METHOD OF SOLVING AN LPP

For linear programming problems having only two variables, the set of all feasible solutions can be displayed graphically by determining the feasible region. The points lying within the feasible region satisfy all the constraints. The graphical approach gives an insight into the basic concepts and provides valuable understanding for solving LP problems involving more than two variables algebraically. Problems involving more than two variables cannot be solved graphically.

There are two techniques of solving an LPP by graphical method

(i) Corner point method.

(ii) Iso-profit or Iso-cost method.

We shall discuss only the first technique.

CORNER POINT METHOD

This method is based on a theorem called extreme point theorem.

Extreme Point Theorem: The optimal solution to a linear programming problem, if it exists, occurs at an extreme point (corner) of the feasible region.

The collection of all feasible solutions to an LPP constitutes a convex set whose extreme points correspond to the basic feasible solutions.

Working procedure to solve an LPP graphically:

1. Formulate the given problem as an LPP.
2. Plot the constraints and shade the common region that satisfies all the constraints simultaneously. The shaded area is called the feasible region.
3. Determine the coordinates of each corner of the feasible region.
4. Find the value of the objective function \( Z = ax + by \) at each corner point.

Let \( M \) and \( m \) denote respectively the largest and the smallest values of \( Z \) at the corner points.
5. When the feasible region is bounded, \( M \) and \( m \) are the maximum and the minimum values of \( Z \).

6. When the feasible region is unbounded, we have:
   (a) \( M \) is the maximum value of \( Z \), if the open half plane determined by \( Z > M \) i.e., \( ax + by > M \) has no point in common with the feasible region. Otherwise \( Z \) has no maximum value.
   (b) \( m \) is the minimum value of \( Z \), if the open half plane determined by \( Z < m \) i.e., \( ax + by < m \) has no point in common with the feasible region. Otherwise \( Z \) has no minimum value.

### ILLUSTRATIVE EXAMPLES

**Example 1.** Solve the following linear programming problem graphically:

Maximize \( Z = 2x + 3y \)

subject to the constraints

\[
\begin{align*}
2x + y & \leq 10 \\
2x + y & \leq 14
\end{align*}
\]

\( x, y \geq 0 \)

**Sol.** Mathematical formulation of LPP is already given. The non-negativity constraints \( x \geq 0, y \geq 0 \) imply that the feasible region is in first quadrant only. Plot each constraint by first treating it as a linear equation and then using the inequality condition of each constraint, mark the feasible region as shown in the figure. The feasible region is shown by the shaded area OABC in the figure.

Since the optimal value of the objective function occurs at one of the corners of the feasible region, we determine their coordinates.

Clearly \( O = (0, 0), \ A = (7, 0), \ B = (6, 2), \ C = (0, 5) \)

Now we find the value of the objective function \( Z = 2x + 3y \) at each corner point:

| Corner Point | Coordinates \((x, y)\) | Value of Objective Function 
\( Z = 2x + 3y \) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( O )</td>
<td>((0, 0))</td>
<td>(0 + 0 = 0)</td>
</tr>
<tr>
<td>( A )</td>
<td>((7, 0))</td>
<td>(2(7) + 3(0) = 14)</td>
</tr>
<tr>
<td>( B )</td>
<td>((6, 2))</td>
<td>(2(6) + 3(2) = 18)</td>
</tr>
<tr>
<td>( C )</td>
<td>((0, 5))</td>
<td>(2(0) + 3(5) = 15)</td>
</tr>
</tbody>
</table>

The maximum value of \( Z \) is 18 at \( B(6, 2) \). Hence the optimal solution to the given LP problem is

\( x = 6, \ y = 2 \) and Max. \( Z = 18 \)

**Remark.** The coordinates of \( B \) are obtained by solving \( x + 2y = 10 \) and \( 2x + y = 14 \).

### Linear Programming

**Example 2.** Solve the following LP problem by the graphical method:

Maximize \( Z = 20x_1 + 10x_2 \)

subject to the constraints

\[
\begin{align*}
x_1 + 2x_2 & \leq 40 \\
3x_1 + x_2 & \leq 30 \\
4x_1 + 3x_2 & \leq 60 \\
x_1, x_2 & \geq 0
\end{align*}
\]

**Solution.** Mathematical formulation of LP is already given. The non-negativity constraints \( x_1, x_2 \geq 0 \) imply that the feasible region is in first quadrant only. Plot each constraint by first treating it as a linear equation and then using the inequality condition of each constraint, mark the feasible region as shown in the figure. The feasible region is shown by the shaded area ABCD in the figure.

Since the optimal value of the objective function occurs at one of the corners of the feasible region, we determine their coordinates.

Clearly \( A = (15, 0), \ B = (40, 0), \ C = (5, 18), \ D = (6, 12) \)

Now we find the value of the objective function \( Z = 20x_1 + 10x_2 \) at each corner point:

| Corner Point | Coordinates \((x, y)\) | Value of Objective Function 
\( Z = 20x_1 + 10x_2 \) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>((15, 0))</td>
<td>(20(15) + 10(0) = 300)</td>
</tr>
<tr>
<td>( B )</td>
<td>((40, 0))</td>
<td>(20(40) + 10(0) = 800)</td>
</tr>
<tr>
<td>( C )</td>
<td>((5, 18))</td>
<td>(20(5) + 10(18) = 280)</td>
</tr>
<tr>
<td>( D )</td>
<td>((6, 12))</td>
<td>(20(6) + 10(12) = 240)</td>
</tr>
</tbody>
</table>

The maximum value of \( Z \) is 240 at \( D(6, 12) \). Hence the optimal solution to the given LP problem is

\( x_1 = 6, \ x_2 = 12 \) and Min. \( Z = 240 \)

**Example 3.** A manufacturer has 3 machines installed in his factory. Machines I and II are capable of being operated for at most 12 hours, whereas machine III must be operated at least 5 hours a day. He produces only two items, each requiring the use of the three machines. The number of hours required for producing 1 unit of each of the items A and B on the three machines are given in the following table:

<table>
<thead>
<tr>
<th>Items</th>
<th>Number of hours required on the machines</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
</tr>
</tbody>
</table>

The coordinates of \( B \) are obtained by solving \( x + 2y = 10 \) and \( 2x + y = 14 \).
He makes a profit of \( \text{Rs} \) 60 on item A and \( \text{Rs} \) 40 on item B. Assuming that he can sell all that he produces, how many of each item should be produced so as to maximize his profit. Solve the LP problem graphically.

Sol. Suppose the manufacturer produces \( x \) and \( y \) units of items A and B respectively. His objective is to maximize the total profit = \( \text{Rs} \) (60x + 40y).

- Objective function is given by \( Z = 60x + 40y \)
- Machine hour constraints are:
  - Machine I: \( x + 2y \leq 12 \)
  - Machine II: \( 2x + y \leq 12 \)
  - Machine III: \( x + \frac{5}{4}y \geq 5 \) or \( 4x + 5y \geq 20 \)

Non-negativity constraints are \( x, y \geq 0 \)

(since it makes no sense to assign negative values to \( x \) and \( y \)).

Thus, the mathematical formulation of the LP problem is:

\[
\text{Maximize } Z = 60x + 40y \\
\text{subject to the constraints: } x + 2y \leq 12 \\
2x + y \leq 12 \\
4x + 5y \geq 20 \\
x, y \geq 0
\]

The feasible region is shown by the shaded area ABCDE in the figure.

Since the optimal value of the objective function occurs at one of the corners of the feasible region, we determine their coordinates.

Here \( A = (5, 0) \), \( B = (6, 0) \), \( C = (4, 4) \), \( D = (0, 6) \), \( E = (0, 4) \)

Now we find the value of the objective function \( Z = 60x + 40y \) at each corner point.

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Coordinates ((x, y))</th>
<th>Value of Objective Function (Z = 60x + 40y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(5, 0)</td>
<td>(60(5) + 40(0) = 300)</td>
</tr>
<tr>
<td>B</td>
<td>(6, 0)</td>
<td>(60(6) + 40(0) = 360)</td>
</tr>
<tr>
<td>C</td>
<td>(4, 4)</td>
<td>(60(4) + 40(4) = 400)</td>
</tr>
<tr>
<td>D</td>
<td>(0, 6)</td>
<td>(60(0) + 40(6) = 240)</td>
</tr>
<tr>
<td>E</td>
<td>(0, 4)</td>
<td>(60(0) + 40(4) = 160)</td>
</tr>
</tbody>
</table>

The maximum value of \( Z \) is 400 at \( C(4, 4) \).

- The manufacturer should produce \( x = 4 \) units of item A and \( y = 4 \) units of item B to get the maximum profit of \( \text{Rs} \) 400.

**Example 4:** A company makes two kinds of leather belts. A and B. Belt A is a high quality belt and B is of lower quality. The respective profits are \( \text{Rs} \) 4 and \( \text{Rs} \) 3 per belt. Each belt of type A requires twice as much time as a belt of type B, and, if all belts were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckles a day available for belt B. What should be the daily production of each type of belt?

Sol. Let \( x \) and \( y \) denote respectively the number of belts of type A and type B produced per day.
The objective is to maximize the profit $Z$ (in rupees) given by

$$ Z = 4x + 3y $$

Since the rate of producing type B belts is 1000 per day, the total time taken to produce $y$ belts of type B is $\frac{y}{1000}$.

Also, each belt of type A requires twice as much time as a belt of type B, the rate of producing type A belts is 500 per day and the total time taken to produce $x$ belts of type A is $\frac{x}{500}$.

- The time constraint is $\frac{x}{500} + \frac{y}{1000} \leq 1$
- The constraint imposed by supply of leather is $x + y \leq 800$
- The constraint imposed by supply of fancy buckles is $x \leq 400$
- The constraint imposed by supply of buckles for type B belts is $y \leq 700$
- Since the number of belts cannot be negative, we have non-negativity constraints $x \geq 0$, $y \geq 0$

- The mathematical formulation of the problem is:

Maximize $Z = 4x + 3y$

subject to the constraints

- $2x + y \leq 1000$
- $x + y \leq 800$
- $x \leq 400$
- $y \leq 700$
- $x, y \geq 0$

The feasible region is shown by the shaded area OABCDE in the figure.

If the optimal value of the objective function occurs at one of the corners of the feasible region, we determine their coordinates.

Here $O = (0,0)$, $A = (400,0)$, $B = (400, 200)$, $C = (200, 600)$, $D = (100, 700)$, $E = (0, 700)$

Now we find the value of the objective function $Z = 4x + 3y$ at each corner point.

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Coordinates</th>
<th>Value of Objective Function $Z = 4x + 3y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>(400, 0)</td>
<td>4000</td>
</tr>
<tr>
<td>B</td>
<td>(400, 200)</td>
<td>2000</td>
</tr>
<tr>
<td>C</td>
<td>(200, 600)</td>
<td>2600</td>
</tr>
<tr>
<td>D</td>
<td>(100, 700)</td>
<td>2500</td>
</tr>
<tr>
<td>E</td>
<td>(0, 700)</td>
<td>2100</td>
</tr>
</tbody>
</table>

The maximum value of $Z$ is 2600 at C(200, 600).

Example 5. Solve the following linear programming problem graphically:

Minimize $Z = 3x + 5y$

subject to the constraints

- $x \leq 4$
- $y \leq 5$
- $x \geq 0$, $y \geq 0$

Sol. Here the feasible region of the LPP is the line segment AB with A(4, 2) and B(1, 5). These are the corner points of the feasible region.

At A(4, 2), $Z = 3(4) + 5(2) = 22$

At B(1, 5), $Z = 3(1) + 5(5) = 28$

The minimum value of $Z$ is 22, which occurs at A(4, 2).

Hence, optimal solution is $x = 4$, $y = 2$ and optimal value is 22.

Example 6. A person consumes two types of food, A and B, everyday to obtain 8 units of protein, 12 units of carbohydrates and 9 units of fat which is his daily minimum requirement.

1 kg of food A contains 2, 6, 1 units of protein, carbohydrates and fat, respectively. 1 kg of food B contains 1, 1 and 3 units of protein, carbohydrates and fat, respectively. Food A costs $5 per kg while food B costs $7 per kg. Form an LPP to find how many kgs of each food should he buy daily to minimize his cost of food and still meet minimal nutritional requirements and solve it.

Sol. Suppose the person buys $x$ kg of food A and $y$ kg of food B daily. Total cost of food (in rupees) is given by

$$ Z = 5x + 7y $$

The objective is to minimize $Z$.

- $x$ kg of food A contains $2x$ units of protein.
- $y$ kg of food B contains $y$ units of protein.

Since minimum requirement of protein is 8 units, therefore, protein constraint is

$$ 2x + y \geq 8 $$

Similarly

$$ 6x + y \geq 12 $$

Also

$$ x, y \geq 0 $$

- The mathematical formulation of the problem is:

Minimize $Z = 5x + 7y$

subject to the constraints

- $2x + y \geq 8$
- $6x + y \geq 12$
The feasible region is shown shaded in the figure. It is unbounded. The extreme points of the feasible region are:

**A** = (9, 0), **B** = (3, 2), **C** = (1, 6), **D** = (0, 12)

Now we find the value of the objective function \( Z = 8x + 5y \) at each corner point.

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Coordinates (x, y)</th>
<th>Value of Objective Function ( Z = 8x + 5y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(9, 0)</td>
<td>8(9) + 5(0) = 72</td>
</tr>
<tr>
<td>B</td>
<td>(3, 2)</td>
<td>8(3) + 5(2) = 34</td>
</tr>
<tr>
<td>C</td>
<td>(1, 6)</td>
<td>8(1) + 5(6) = 38</td>
</tr>
<tr>
<td>D</td>
<td>(0, 12)</td>
<td>8(0) + 5(12) = 60</td>
</tr>
</tbody>
</table>

The minimum value of \( Z \) is 34 at **B**(3, 2).

But this value is doubtful. It is yet to be confirmed. Now we draw the graph of \( Z < m \) i.e., \( 8x + 5y < 34 \). Since the half plane determined by \( Z < m \) has no point in common with the feasible region, \( m = 34 \) is the minimum value of \( Z \).

The person should buy 3 kg of food A and 2 kg of food B at the minimum cost of Rs. 34.

### II. Unbounded Solutions.

Some linear programming problems have unbounded feasible region so that the variables can take any value in the unbounded region without violating any constraint. If we wish to maximize the objective function \( Z \), then for any value of \( Z \) we can find a feasible solution with a greater value of \( Z \). Such problems are said to have unbounded solutions.

**Example.** Maximize \( Z = 2x + 3y \) subject to the constraints \( x + y \geq 3 \), \( x - y \leq 1 \), \( x, y \geq 0 \).

**Sol.** The feasible region, shown shaded in the figure, is unbounded. The corner points of the feasible region are A(0, 3) and B(2, 1).

\[
Z(A) = 2(0) + 3(3) = 15 \\
Z(B) = 2(2) + 3(1) = 9
\]
Since the given LP problem is of maximization. There are infinite number of points in the feasible region where the value of the objective function is more than this value \( Z(A) = 5 \). Both the variables \( x \) and \( y \) can be made arbitrarily large and there is no limit to the value of \( Z \). Hence the problem has unbounded solutions.

### III. Multiple Optimal Solutions.

When there exist more than one points in the feasible region such that the objective function \( Z \) has the same optimal value, say \( k \), then each such point corresponds to an optimal solution. The LPP has multiple solutions. Each of such optimal solutions is called multiple optimal solution or an alternative optimal solution.

The following two conditions must be satisfied for multiple optimal solutions to exist:

(i) the objective function must be parallel to a constraint which forms the boundary of the feasible region.

(ii) the constraint must form a boundary of the feasible region in the direction of the optimal movement of the objective function.

### Example. Solve the following problem graphically.

**Maximize**  
\[ Z = 2.5x + y \]

subject to the constraints  
\[ 3x + 5y \leq 15 \]
\[ 5x + 2y \leq 10 \]
\[ x, y \geq 0. \]

**Sol.** The feasible region is shown by the shaded area OABC in the figure.

We give a constant value 2.5 to \( Z \) and draw the line \( 2.5x + y = 2.5 \), shown by \( P_1Q_1 \). Give another value 10 to \( Z \) and draw the line \( 2.5x + y = 10 \), shown by \( P_2Q_2 \). Moving \( P_1Q_1 \) parallel to itself towards \( Z \) increasing i.e., towards \( P_2Q_2 \), as far as possible, until the farthest point \( B \) within the feasible region is touched by the line, shown by \( P_1Q_1 \). Clearly \( P_1Q_1 \) contains the line segment \( AB \) which corresponds to the constraint \( 5x + 2y \leq 10 \) and forms the boundary of the feasible region. Solving \( 3x + 5y = 15 \) and \( 5x + 2y = 10 \), we have \( B = \left( \frac{20}{19}, \frac{45}{19} \right) \) and

\[ Z(B) = \frac{5}{2} \cdot \frac{20}{19} + \frac{45}{19} = 5. \]

Also \( A = (2, 0) \) and

\[ Z(A) = 2.5 \times 2 + 0 = 5. \]

All points on the line segment \( AB \) give the same optimal value \( Z = 5 \). The optimal value is unique but there are infinite number of optimal solutions. Every point on \( AB \) corresponds to an optimal solution.

Here the objective function has the same slope as the constraint line \( 5x + 2y = 10 \).

### IV. Redundancy.

A constraint in a given LPP is said to be redundant if the feasible region of the problem is not affected by deleting that constraint.
is a convex set. Find the extreme points of this set. Hence solve LPP graphically:

7. Maximize \( Z = 4x_1 + 3x_2 \) subject to the constraints given in S.

\[
\begin{align*}
\text{subject to the constraints} \quad & x_1 + y_2 \leq 40 \\
& 3x_1 + y_2 \leq 90 \\
& 4x_1 + y_2 \leq 60 \\
& x_1, x_2 \geq 0 \\
& x_1, x_2 \geq 0.
\end{align*}
\]

(M.D.U. Dec. 2006)

8. (a) Maximize \( Z = 2x_1 + 3x_2 \) subject to the constraints \( x_1 + x_2 \leq 30 \)

\[
\begin{align*}
& x_1 \geq 3 \\
& 0 \leq x_2 \leq 12 \\
& 0 \leq x_1 \leq 20 \\
& x_1 - x_2 \geq 0 \\
& x_1, x_2 \geq 0.
\end{align*}
\]

(M.D.U. Dec. 2005)

(b) Find a geometrical interpretation and solution as well for the following L.P.P problem:

Maximize \( Z = 5x_1 + 3x_2 \)

subject to restrictions \( x_1 + 2x_2 \leq 2000 \)

\[
\begin{align*}
& x_1 + x_2 \leq 1500 \\
& x_1 \leq 600 \\
& x_1, x_2 \geq 0.
\end{align*}
\]


(c) Solve the following L.P.P. graphically:

Maximize \( Z = 6x_1 + 3x_2 \)

subject to

\[
\begin{align*}
& 4x_1 + 5x_2 \leq 1000 \\
& 5x_1 + 2x_2 \leq 1000 \\
& 3x_1 + 12x_2 \leq 1200 \\
& x_1, x_2 \geq 0.
\end{align*}
\]

(M.D.U. May 2011)

9. (a) A housewife wishes to mix two types of foods X and Y in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food X costs Rs. 60 per kg and food Y costs Rs. 80 per kg. Food X contains 3 units per kg of vitamin A and 5 units per kg of vitamin B while food Y contains 4 units per kg of vitamin A and 2 units per kg of vitamin B. Formulate the above problem as an L.P.P to minimize the cost of the mixture.

(M.D.U. Dec. 2006)

(b) A housewife wishes to mix together two kinds of food: X and Y, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg of food is given below:

<table>
<thead>
<tr>
<th></th>
<th>Vitamin A</th>
<th>Vitamin B</th>
<th>Vitamin C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food X</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Food Y</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

One kg of food X costs Rs. 6 and one kg of food Y costs Rs. 10. Formulate the above problem as an L.P.P to find the least cost of the mixture which will produce the diet and solve it.

(c) A manufacturer of furniture makes two products: chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by selling one chair is Rs. 10 and from a table is Rs. 50. What should be the daily production plan of the factory?


10. A retired person wants to invest an amount of up to Rs. 20,000. His broker recommends investing in two types of bonds A and B. Bond A yielding 10% return on the amount invested and bond B yielding 15% return on the amount invested. After some consideration, he decides to invest at least Rs. 5000 in bond A and no more than Rs. 8000 in bond B. He also wants to invest at least as much in bond A as in bond B. What should his broker suggest if he wants to maximize his return on investments? Formulate the above problem as an L.P.P and solve it.

(M.D.U. May 2006)

11. A factory manufactures two types of screws A and B. Each type requires the use of two machines, one automatic and one hand operated. It takes 4 minutes on the automatic and 6 minutes on the hand operated machines to manufacture a package of screws A, while it takes 5 minutes on the automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs. 1 and a package of screws B at a profit of Rs. 1. Assuming that he can sell all the screws he can manufacture, how many packages of each type should the factory owner produce in a day in order to maximize his profit? Determine the maximum profit.

12. A firm manufactures two products X and Y, each requiring the use of three machines M, M, and M. The time required for each product in hours and total time available in hours on each machine are as follows:

<table>
<thead>
<tr>
<th>Machine</th>
<th>Product X</th>
<th>Product Y</th>
<th>Available time (in hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>2</td>
<td>1</td>
<td>70</td>
</tr>
<tr>
<td>M2</td>
<td>1</td>
<td>1</td>
<td>40</td>
</tr>
<tr>
<td>M3</td>
<td>1</td>
<td>3</td>
<td>50</td>
</tr>
</tbody>
</table>

If the profit is Rs. 40 per unit for product X and Rs. 60 per unit for product Y, how many units of each product should be manufactured to maximize profit?

13. A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 10 units of vitamin B. Food I contains 2 units/kg of vitamin A and 3 units/kg of vitamin B, while food II contains 1 unit/kg of vitamin A and 2 units/kg of vitamin B. It costs Rs. 5 per kg to purchase food I and Rs. 7 per kg to purchase food II. Determine the minimum cost of such a mixture.

14. (a) A company has two depots A and B, with capacities of 3000 L and 4000 L, respectively. The company is to supply oil to three petrol pumps C, D, and E whose requirements are 4500 L, 3000 L, and 3500 L, respectively. The distances (in km) between the depots and the petrol pumps are given in the following table:

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Distance (in km)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>F</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Assuming that the transportation cost per km is Rs. 1 per L, how should the delivery be scheduled in order that the transportation cost is minimum?

(b) There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B, and C. The weekly requirements of the depots are respectively 5, 5, and 4 units of the commodity. The company plans to manufacture this commodity at P and Q. Formulate the transportation problem as an L.P.P and solve it.
the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>100</td>
<td>100</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>Q</td>
<td>100</td>
<td>120</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

How many units should be transported from each factory to each depot in order that the transportation cost is minimum? What will be the minimum transportation cost?

15. A manufacturer of patent medicines is preparing a production plan on medicines A and B. There are sufficient raw materials available to make 20,000 bottles of A and 40,000 bottles of B, but there are only 45,000 bottles into which neither of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of A and 1 hour to prepare enough material to fill 1000 bottles of B. There are 66 hours available for this operation. The profit is Rs 8 per bottle for A and Rs 7 per bottle for B. How should the manufacturer schedule his production in order to maximize his profit?

16. A small manufacturer has employed 5 skilled men and 10 semi-skilled men and makes an article in two qualities, a deluxe model and an ordinary model. The making of a deluxe model requires 2 hours of work by a skilled man and 2 hours of work by a semi-skilled man. The ordinary model requires 1 hour by a skilled man and 3 hours by a semi-skilled man. By union rule, no man may work for more than 8 hours per day. The manufacturer gains Rs 15 on a deluxe model and Rs 10 on an ordinary model. How many of each type should be made in order to maximize his total daily profit?

17. Minimize \( Z = 3x_1 + 2x_2 \)
subject to the constraints:
- \( 5x_1 + x_2 \geq 10 \)
- \( x_1 + x_2 \leq 6 \)
- \( x_1 + 4x_2 \leq 12 \)
- \( x_1, x_2 \geq 0 \)

18. A firm plans to purchase at least 200 quintals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain at least 100 quintals of X-metal and not more than 35 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of weight in the scrap supplied by A and B is given below:

<table>
<thead>
<tr>
<th>Metals</th>
<th>Supplier A</th>
<th>Supplier B</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>26%</td>
<td>75%</td>
</tr>
<tr>
<td>Y</td>
<td>10%</td>
<td>20%</td>
</tr>
</tbody>
</table>

The price of A's scrap is Rs 200 per quintal and that of B is Rs 400 per quintal. The firm wants to determine the quantities that it should buy from the two suppliers so that total cost is minimized. Formulate this as an LPP and solve it.

19. G.J. Breweries Ltd. has two bottling plants, one located at 'G' and the other at 'J'. Each plant produces three drinks: whisky, beer, and brandy, named A, B, and C, respectively. The number of bottles produced per day are as follows:

<table>
<thead>
<tr>
<th>Drink</th>
<th>Plant at</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>G</td>
</tr>
<tr>
<td>Whisky</td>
<td>1,500</td>
</tr>
<tr>
<td>Beer</td>
<td>3,000</td>
</tr>
<tr>
<td>Brandy</td>
<td>2,000</td>
</tr>
</tbody>
</table>

A market survey indicates that during the month of July there will be a demand for 20,000 bottles of whisky, 40,000 bottles of beer and 44,000 bottles of brandy. The operating cost per day run is Rs 40 per plant. For plants G and J are 600 and 100 monetary units. For how many days should each plant be run graphically. Verify that the following problems have no feasible solution.

(a) Maximize \( Z = 15x + 20y \)
subject to the constraints:
- \( x + y \geq 12 \)
- \( 6x + 9y \leq 54 \)
- \( 15x + 10y \leq 90 \)

(b) Maximize \( Z = 4x + 3y \)
subject to the constraints:
- \( x + y \leq 3 \)
- \( 2x - y \leq 3 \)
- \( x \geq 4 \)
- \( x, y \geq 0 \)

Verify that the following problems have multiple solutions.

(a) Minimize \( Z = 12x + 8y \)
subject to the constraints:
- \( 3x + 6y \geq 36 \)
- \( 4x + 3y \geq 24 \)
- \( x + y \leq 15 \)
- \( x, y \geq 0 \)

(b) Maximize \( Z = 6x + 4y \)
subject to the constraints:
- \( x + y \geq 5 \)
- \( 3x + 2y \geq 12 \)
- \( x, y \geq 0 \)

Verify that the following problem has an unbounded solution:

Maximize \( Z = 20x + 30y \)
subject to the constraints:
- \( 5x + 2y \geq 50 \)
- \( 2x + 6y \geq 20 \)
- \( 4x + 3y \geq 60 \)
- \( x, y \geq 0 \)

Graph the feasible region of the following problem and identify a redundant constraint.

Minimize \( Z = 6x + 10y \)
subject to the constraints:
- \( x \geq 6 \)
- \( y \geq 2 \)
- \( 2x + y \geq 10 \)
- \( x, y \geq 0 \)

Answers

1. \( x_1 = \frac{8}{5} \), \( x_2 = \frac{12}{5} \); Max. \( Z = 24.8 \)
2. \( x = 18 \), \( y = 12 \); Max. \( Z = 72 \)

3. \( x_1 = 20 \), \( x_2 = \frac{45}{19} \); Max. \( Z = 23.5 \)
4. \( x_1 = 2 \), \( x_2 = 3 \); Min. \( Z = 84 \)

5. \( x_1 = 1 \), \( x_2 = 5 \); Min. \( Z = 13 \)
6. \( x_1 = 50 \), \( x_2 = 0 \); Max. \( Z = 200 \)

7. \( x = 6 \), \( y = 12 \); Min. \( Z = 240 \)
8. (a) \( x_1 = 18 \), \( x_2 = 12 \); Max. \( Z = 72 \)
(b) \( x_1 = 1000 \), \( x_2 = 500 \); Max. \( Z = 5500 \)

(c) \( x_1 = \frac{5000}{17} \), \( x_2 = \frac{1000}{17} \); Max. \( Z = 1058.82 \)
(d) No maximum
9. (a) Minimum cost = \( \text{Rs} \ 160 \)
   (b) 2 kg of food X, 4 kg of food Y, least cost = \( \text{Rs} \ 52 \)
   (c) Number of chairs = 0, Number of tables = 165, i.e., 16 in 5 days
   (d) \( \text{Rs} \ 1200 \) in bond A, \( \text{Rs} \ 400 \) in bond B; Maximum return = \( \text{Rs} \ 2400 \)
10. (a) \( \text{Rs} \ 2000 \) in bond A, \( \text{Rs} \ 8000 \) in bond B; Maximum return = \( \text{Rs} \ 2400 \)
11. Screw A = 30 packages, Screw B = 20 packages; Maximum profit = \( \text{Rs} \ 41 \)
12. Product X = 15, Product Y = 25; Maximum profit = \( \text{Rs} \ 2100 \)
13. Food I = 2 kg, Food II = 4 kg; Minimum cost = \( \text{Rs} \ 38 \)
14. (a) From A: 500 I, 3000 I, 3500 I to D, E, F respectively; From B: 400 I, 0 I, 0 I to D, E, F respectively; Minimum cost = \( \text{Rs} \ 44,000 \)
   (b) From A: 6, 5, 3 units to A, B, C respectively; From Q: 0, 0, 0 units to A, B, C respectively
   Minimum transportation cost = \( \text{Rs} \ 1500 \)
15. 10500 bottles of A, 34500 bottles of B; Maximum profit = \( \text{Rs} \ 325500 \)
16. Delux model = 10, ordinary model = 20; Maximum profit = \( \text{Rs} \ 350 \)
   (Hint: 5 skilled men cannot work for more than 5 \( \times \) 8 = 40 hours and 10 semi-skilled men cannot work for more than 10 \( \times \) 8 = 80 hours.)
17. \( x_1 = 1, x_2 = 5 \); Min. Z = 13
18. Supplier A = 100 quintals, supplier B = 100 quintals; Minimum cost = \( \text{Rs} \ 60,000 \)
19. Plant at G = 12 days, Plant at J = 4 days; Minimum cost = 8,800 monetary units.
20. \( 2x + y \geq 10 \).

### 7.7. INTRODUCTION

The linear programming problems discussed so far are concerned with two variables, the solution of which can be found out easily by the graphical method. But most real-life problems when formulated as an LP model involve more than two variables and many constraints. Thus, there is a need for a method other than the graphical method. The most popular non-graphical method of solving an LP problem is called the simplex method. This method, developed by George B. Dantzig in 1947, is applicable to any problem that can be formulated in terms of linear objective function subject to a set of linear constraints. There are no theoretical restrictions placed on the number of decision variables or constraints. The development of computers has further made it easy for the simplex method to solve large-scale LP problems very quickly.

The concept of the simplex method is similar to the graphical method. For LP problems with several variables, the optimal solution lies at a corner point of the many-faced, multi-dimensional figure, called an \( n \)-dimensional polyhedron. The simplex method examines the corner points in a systematic manner. It is a computational routine of repeating the same set of steps over and over until an optimal solution is reached. For this reason, it is known as an iterative method. As we move from one iteration to the other, the method improves the value of the objective function and achieves optimal solution in a finite number of iterations.

### 7.8. SOME USEFUL DEFINITIONS

(i) **Slack Variable.** A variable added to the left-hand side of a less than or equal to constraint to convert the constraint into an equality is called a slack variable.

(ii) **Surplus Variable.** A variable subtracted from the left-hand side of a greater than or equal to constraint to convert the constraint into an equality is called a surplus variable.

(iii) **Basic Solution.** For a system of \( m \) simultaneous linear equations in \( n \) variables \((n > m)\), a solution obtained by setting \((n - m)\) variables equal to zero and solving for the remaining \( m \) variables for a unique solution is called a basic solution. The \((n - m)\) variables set equal to zero in any solution are called non-basic variables. The other \( m \) variables are called basic variables.

(iv) **Degenerate and Non-degenerate Solution.** If one or more of the basic variables in the basic feasible solution are zero, then it is called a degenerate solution. If all the variables in the basic feasible solution are positive, then it is called a non-degenerate solution.

**Example.** Consider the following LPP:

Maximize \( Z = 5x + 8y \)

subject to the constraints

\[
\begin{align*}
2x + y & \leq 4 \\
x + y & \leq 6 \\
x, y & \geq 0.
\end{align*}
\]

Sol. Introducing slack variables \( s_1, s_2 \) to convert less than or equal to constraints into equalities, we get

\[
\begin{align*}
x + y + s_1 & = 4 \\
2x + y + s_2 & = 6
\end{align*}
\]

We have two equations \((m = 2)\) in four variables \((n = 4)\). For a basic solution, we put \( n - m = 4 - 2 = 2 \) variables equal to zero and solve for the other two. The various possibilities are shown in the following table:
7.9. STANDARD FORM OF AN LPP

The standard form of an LPP should have the following characteristics:

(i) Objective function should be of maximization type.

(ii) All constraints should be expressed as equations by adding slack or surplus variables, one for each constraint.

(iii) The right-hand side of each constraint should be non-negative. If it is negative, then to make it positive, we multiply both sides of the constraint by (-1), changing ≤ to ≥ and vice versa.

(iv) All variables are non-negative.

Thus, the standard form of an LPP with \( n \) variables and \( m \) constraints is:

Maximize \( Z = c_1x_1 + c_2x_2 + \ldots + c_mx_m + s_0 \), subject to the constraints

\[
\begin{align*}
\sum a_{1i}x_i + s_1 &= b_1 \\
\sum a_{2i}x_i + s_2 &= b_2 \\
\vdots & \vdots \\
\sum a_{mi}x_i + s_m &= b_m 
\end{align*}
\]

and \( x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m \geq 0 \).
Sol. Here $x_1$, $x_3$ are the only two decision variables and $x_3$, $x_4$, $x_5$ are the slack/surplus variables. Also $x_2$ is unrestricted, so let $x_2 = x_2^* - x_2^*$, where $x_2^* - x_2^* = 0$. Since right hand side of second constraint is negative, we multiply throughout by $(-1)$.

The given problem in standard form is:

Maximize $Z = -5x_1 - 8x_2 + 8x_3 + 0x_4 + 0x_5$

subject to the constraints

$3x_1 + 7x_2^* - 7x_3^* + x_3 = 9$
$-4x_1 + 2x_2^* - 2x_3^* + x_4 = 15$
$2x_1 - 3x_2^* + 3x_3^* + x_5 = 8$
$x_1, x_2^*, x_3^*, x_3, x_4, x_5 \geq 0.$

[Min. $Z = - Max. Z^*$]

**EXERCISE 7.4**

1. Find all the basic solutions of the following system of equations identifying in each case the basic and non-basic variables:

$2x_1 + x_2 + 4x_3 = 11, \ 3x_1 + x_3 + 5x_4 = 14.$

Investigate whether the basic solutions are degenerate basic solutions or not. Hence find the basic feasible solution of the system.

2. Obtain all the basic solutions to the following system of linear equations:

$x_1 + 2x_2 + x_4 = 4, \ 2x_1 + x_3 + 5x_4 = 5$

Which of them are feasible? Point out non-degenerate basic solutions, if any.

3. Show that the following system of linear equations has two degenerate feasible solutions and the non-degenerate basic solution is not feasible.

$2x_1 - x_2 - x_3 = 0, \ 3x_1 + 2x_2 + x_3 = 3.$

4. Find all the basic solutions to the following problem:

Maximize $Z = x_1 + 3x_2 + 3x_3$

subject to the constraints

$x_1 + 2x_2 + 3x_3 = 4$
$2x_1 + 3x_2 + 5x_3 = 7$
$x_1, x_2, x_3 \geq 0$

Which of the basic solutions are (i) non-degenerate basic feasible (ii) optimal basic feasible?

5. Find an optimal solution to the following LPP by computing all basic solutions and then finding one that maximizes the objective function:

Maximize $Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$

subject to the constraints

$2x_1 + 3x_2 - x_3 + 4x_4 = 8$
$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$
$x_1, x_2, x_3, x_4 \geq 0$

6. Express the following LP problems in the standard form:

(i) Maximize $Z = 3x_1 + 4x_2 + 5x_3 + 0x_4 + 0x_5$

subject to the constraints

$2x_1 + 7x_2 + x_3 \leq 10$
$5x_1 + 9x_2 + 4x_3 \geq 20$
$8x_1 + 15x_2 \leq 30$
$x_1, x_2, x_3 \geq 0$

(ii) Maximize $Z = x_1 + 7x_2 + 2x_3 - 2x_4 + 0x_5 + 0x_6$

subject to the constraints

$3x_1 + 2x_2 + x_3^* - x_4^* + x_5 = 8$
$5x_1 + 7x_2 - x_5 = 14$
$4x_2 + 3x_3^* - 3x_4^* + x_6 = 12$
$x_1, x_2, x_3^*, x_4^*, x_5, x_6 \geq 0$

**ANSWERS**

1. | Non-basic variables | Basic variables | Basic solutions |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3^*$</td>
<td>$x_1, x_2$</td>
<td>$x_3 = 3, \ x_2 = 5$</td>
</tr>
<tr>
<td>$x_1^*$</td>
<td>$x_2, x_3$</td>
<td>$x_2^* = 1, \ x_3 = 3$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$x_2, x_3$</td>
<td>$x_2 = \frac{1}{2}, \ x_3 = \frac{5}{2}$</td>
</tr>
</tbody>
</table>

First and third are basic feasible solutions which are also non-degenerate basic solutions.

2. $x_1 = 2, \ x_2 = 1$: feasible, non-degenerate

$\frac{5}{3}, \ x_3 = \frac{2}{3}$: feasible, non-degenerate

$x_1 = 5, \ x_3 = -1$: non-feasible.

3. $x_1 = 2, x_2 = 1; \ x_1 = x_3 = 1; \ x_2 = 1, x_3 = 2$  
$(x_3 = 0) \quad (x_2 = 0) \quad (x_1 = 0)$

(i) First two solutions are non-degenerate basic feasible solutions

(ii) First solution is optimal basic feasible and Max. $Z = 5$.

5. Optimal basic feasible solution is $x_1 = 0, x_2 = 0, x_3 = 44 \frac{17}{17}, x_4 = 45 \frac{17}{17}$ and Max. $Z = 491 \frac{17}{17}$

6. (i) Maximize $Z = 3x_1 + 4x_2 + 5x_3 + 0x_4 + 0x_5$

subject to the constraints

$2x_1 + 7x_2 + x_3 = 10$
$5x_1 + 9x_2 + 4x_3 = 20$
$8x_1 + 15x_2 = 30$
$x_1, x_2, x_3 \geq 0$

(ii) Maximize $Z = x_1 + 7x_2 + 2x_3 - 2x_4 + 0x_5 + 0x_6$

subject to the constraints

$3x_1 + 2x_2 + x_3^* - x_4^* + x_5 = 8$
$5x_1 + 7x_2 - x_5 = 14$
$4x_2 + 3x_3^* - 3x_4^* + x_6 = 12$
$x_1, x_2, x_3^*, x_4^*, x_5, x_6 \geq 0$
(iii) Maximize \( Z' = -z_1 - 3z_2 - 3z_3 + 0z_4 + 0z_5 + 0z_6 \)
subject to the constraints
\[
\begin{align*}
3x_1 - x_2 + 2x_3 + s_1 &= 7 \\
-2x_1 - 4x_2 + s_2 &= 12 \\
-4x_1 + 3x_2 + 8x_3 + s_3 &= 10 \\
x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0.
\end{align*}
\]

### 7.10. Working Procedure of Simplex Method

Assuming the existence of an initial basic feasible solution, the optimal solution to an LPP by simplex method is obtained as follows:

**Step 1: To express the LPP in the Standard Form**

(i) Formulate the mathematical model of the given LPP.

(ii) If the objective function is to be minimized, then convert it into a maximization problem by using

\[ \text{Min. } Z = - \text{Max. } (-Z) \]

(iii) The right-hand side of each constraint should be non-negative.

(iv) Express all constraints as equations by introducing slack/surplus variables, one for each constraint.

(v) Restate the given LPP in standard form: Maximize \( Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0s_1 + 0s_2 + \ldots + 0s_m \)
subject to the constraints
\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + s_2 &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + s_m &= b_m
\end{align*}
\]
and
\[
\begin{align*}
x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m &\geq 0
\end{align*}
\]

**Step 2: To set up the Initial Basic Feasible Solution**

Take the \( m \) slack/surplus variables \( s_1, s_2, \ldots, s_m \) as the basic variables so that the \( n \) given variables \( x_1, x_2, \ldots, x_n \) are non-basic variables. As such, \( x_1 = x_2 = \ldots = x_n = 0 \) and \( s_1 = b_1, s_2 = b_2, \ldots, s_m = b_m \).

Since each \( b_i \) is non-negative [see Step 1 (iii)], the basic solution is feasible. This basic feasible solution is the starting point of the iterative process. The simplex method then proceeds to solve the LPP by designing and re-designing successively better basic feasible solutions until an optimal solution is obtained.

**Step 3: To set up the Initial Simplex Table**

The above information is conveniently expressed in the tabular form as shown below.

For computational efficiency, the tabular form is better.
9. The last row labelled $C_j = c_j - Z_j$, is called the index row or net evaluation row.

For any column $c_j$, the value for that column written at the top of that column

$-Z_j$ (value for that column)

It should be noted that $C_j$ values are meaningful for non-basic variables only. For a basic variable, $C_j = 0$.

The $C_j$-row represents the net contribution to the objective function that results by introducing one unit of each of the respective column variables. A plus value indicates that a greater contribution can be made by bringing the variable for that column into the solution. A negative value indicates the amount by which this contribution would decrease if one unit of the variable for that column were brought into the solution.

Step 4: Test for Optimality. Examine the entries in the $C_j$-row. If all entries in this row are negative or zero, i.e., if $C_j \leq 0$, then the basic feasible solution is optimal. Any positive entry in the row indicates that an improvement in the value of objective function $Z$ is possible and, hence, we proceed to the next step.

Step 5: To Identify the Incoming and Outgoing Variables. If there is a positive entry in the $C_j$-row, then simplex method shifts from the current basic feasible solution to a better basic feasible solution. For this, we have to replace one current basic variable (called the outgoing or departing variable) by a non-basic variable (called the incoming or entering variable).

Determination of Incoming Variable. The column with the largest positive entry in the $C_j$-row is called the key (or pivot) column (which is shown marked with an arrow ↑). The non-basic variable which will replace a basic variable is the one lying in the key column. Thus the incoming variable is located.

If more than one variable has the same positive largest entry in the $C_j$-row, then any of these variables may be selected arbitrarily as the incoming variable.

Determination of Outgoing Variable. Divide each entry of the solution column (i.e., $x_i$ column) by the corresponding positive entry in the key column. These quotients are written in the last column labelled 'Ratio'. The row which corresponds to the smallest non-negative quotient is called the key (or pivot) row (which is also marked with an arrow →). The departing variable is the corresponding basic variable in this row. The element at the intersection of the key row and key column is called the key (or pivot) element. We place a circle around this element.

If all these ratios are negative or zero, the incoming variable can be made as large as we please without violating the feasibility condition. Hence the problem has an unbounded solution and no further iteration is required.

Step 6: To set up the new simplex table from the current one.

Drop the outgoing variable and introduce the incoming variable along with its associated value under the $c_j$ column.

If the key element is 1, then the key row remains the same in the new simplex table.

If the key element is not 1, then to reduce it to 1, divide each element in the key row (including elements in $x_i$ column) by the key element.

Thus, new key row = old key row

key element

Make all other elements of the key column 0 by subtracting suitable multiples of key row from the other rows. In either words, to change the non-key row, we use the formula:

$\text{Number in new non-key row} = (\text{Number in old non-key row}) - (\text{key column entry}) \times (\text{corresponding number in new key row})$

where 'key column entry' is the entry in this row that is in the key column.

Step 7: Test for Optimality. If there is no positive entry in the $C_j$-row, we have an optimal solution. Otherwise, go to Step 4 and repeat the procedure until all entries in the $C_j$-row are either negative or zero.

ILLUSTRATIVE EXAMPLES

Example 1. Using simplex method

Maximize $Z = 2x_1 + 5x_2$

subject to

$x_1 + 4x_2 \leq 24$

$3x_1 + x_2 \leq 21$

$x_1 + x_2 \leq 9$

$x_1, x_2 \geq 0$.

(M.D.U. Dec. 2007)

Sol. Step 1. To formulate the mathematical model of the problem. We are already given the mathematical model of the LPP. The problem is of maximization type and all $b_i$'s are positive.

Step 2. To express the problem in standard form. Introducing slack variables $s_1, s_2, s_3$ (one for each constraint) the problem in standard form is:

Maximize $Z = 2x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3$

subject to

$x_1 + 4x_2 + s_1 = 24$ \hspace{1cm} (1)

$3x_1 + x_2 + s_2 = 21$ \hspace{1cm} (2)

$x_1 + x_2 + s_3 = 9$ \hspace{1cm} (3)

$x_1, x_2, s_1, s_2, s_3 \geq 0$.

Step 3. To set up the initial basic feasible solution. Since we have 3 equations in 5 variables, a solution is obtained by setting $5 - 3 = 2$ variables equal to zero and solving for the remaining 3 variables. We start with a basic solution by setting $x_1 = x_2 = 0$. Substituting $x_1 = x_2 = 0$ in (1), (2) and (3), we get the basic solution

$s_1 = 24, \quad s_2 = 21, \quad s_3 = 9$

Since $s_1, s_2, s_3$ are all positive, the basic solution is feasible and non-degenerate.

Thus our initial basic feasible solution is

$x_1 = 0, \quad x_2 = 0, \quad s_1 = 24, \quad s_2 = 21, \quad s_3 = 9$

at which $Z = 0$. This initial basic feasible solution is summarized in the following initial simplex table.
### Simplex Table I

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>Basis</th>
<th>Solution $b = x_d$</th>
<th>2</th>
<th>5</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>Ratio $x_d / x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$s_1$</td>
<td>24</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>24 / 4 = 6</td>
</tr>
<tr>
<td>0</td>
<td>$s_2$</td>
<td>21</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>21 / 1 = 21</td>
</tr>
<tr>
<td>0</td>
<td>$s_3$</td>
<td>9</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>9 / 1 = 9</td>
</tr>
<tr>
<td>$Z = 0$</td>
<td>$Z_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$Z = 0$</td>
<td>$Z_2$</td>
<td>2</td>
<td>6</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

*For writing the body matrix (under $x_1, x_2$) and the identity matrix (under $s_1, s_2, s_3$), the left hand sides of equations (1), (2), (3) should be treated as:

- $x_1 + 4x_2 + 1s_1 + 0s_2 + 0s_3 = 0$
- $3x_1 + 1x_2 + 0s_1 + 1s_2 + 0s_3 = 0$
- $1x_1 + 1x_2 + 0s_1 + 0s_2 + 1s_3 = 0$

*The Z-row entries are all equal to zero, i.e., it is the initial simplex table.

**Step 4. Test for Optimality.** Since some entries in $C$-row are positive, the current solution is not optimal. Therefore, an improvement in the value of objective function $Z$ is possible and we proceed to the next step.

**Step 5. To identify the incoming and outgoing variables.** The largest positive entry in the $c_i$-row is 5 and the column in which it appears is headed by $x_2$. Thus $x_2$ is the incoming variable and $x_3$-column is the key column (indicated by $\hat{3}$).

Dividing each entry of the solution column by the corresponding positive entry in the key column, we find that minimum positive ratio is 6 and it appears in the row headed by $s_1$. Thus $s_1$ is the outgoing basic variable and the corresponding row is the key row (indicated by $\rightarrow$). The number at the intersection of the key row and the key column is the key number. Thus 6 is the key number. A circle is placed around this number.

**Step 6. To set up the new simplex table from the current one.** Drop the outgoing variable $s_1$ from the basis and introduce the incoming variable $x_2$. The new basis will contain $x_2, s_2, s_3$ as the basic variables. The coefficient of $x_2$ in the objective function is 5. Therefore the entry in $c_i$-column corresponding to the new basic variable $x_2$ will be 5. Since the key element enclosed in the circle is not 1, divide all elements of the key row by the key element 5 to obtain new values of the elements in this row. Thus the key row 0 $s_1$ 24 1 4 1 0 0 is replaced by the new key row 5 $x_2$ 6 1 4 1 1 0 0

Now we make all other elements of the key column (i.e., $x_2$-column) zero by subtracting suitable multiples of key row from the other rows.

### Simplex Table II

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>Basis</th>
<th>Solution $b = x_d$</th>
<th>2</th>
<th>5</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>Ratio $x_d / x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$x_2$</td>
<td>6</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5 / 1 = 5</td>
</tr>
<tr>
<td>0</td>
<td>$s_2$</td>
<td>15</td>
<td>11</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>15 / 11 = 1.363</td>
</tr>
<tr>
<td>0</td>
<td>$s_3$</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3 / 1 = 3</td>
</tr>
<tr>
<td>$Z = 30$</td>
<td>$Z_1$</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$Z = 30$</td>
<td>$Z_2$</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

$*Z = \sum c_B x_B = 5(6) + 0(15) + 0(3) = 30$
Step 7. Test for optimality. Since all entries in Cj-row are not negative or zero, the current solution is not optimal. Therefore, an improvement in the value of objective function \( Z \) is possible and we repeat steps 5 to 7.

Incoming variable is \( x_1 \) and \( x_1 \)-column is the key-column. \( s_3 \) is the outgoing basic variable and \( s_3 \)-row is the key row. Key number is \( \frac{3}{4} \).

The new basis will contain \( x_2 \), \( s_2 \), \( x_1 \) as the basic variables. Dividing all elements of the key row by the key number \( \frac{3}{4} \), the new key row is
\[
\begin{align*}
2 & \quad x_1 \quad 4 \quad 1 \quad 0 \quad -\frac{1}{3} \quad 0 \quad \frac{4}{3} \\
\end{align*}
\]

Transformation of \( R_1 \). Key column entry in \( R_1 \) is \( \frac{1}{4} \).

\[
R_1(\text{new}) = R_1(\text{old}) - \frac{1}{4} R_1(\text{new})
\]
\[
= 6 - \frac{1}{4}(4) = 5 \\
= \frac{1}{4} \left( \frac{1}{1} \right) = 0 \\
= 1 - \frac{1}{4}(0) = 1 \\
= \frac{1}{4} \left( \frac{1}{1} \right) = \frac{1}{3} \\
= 0 - \frac{1}{4}(0) = 0 \\
= 0 - \frac{1}{4}(0) = 0 \\
\]

Transformation of \( R_2 \). Key column entry in \( R_2 \) is \( \frac{11}{4} \).

\[
R_2(\text{new}) = R_2(\text{old}) - \frac{11}{4} R_2(\text{new})
\]
\[
= 15 - \frac{11}{4}(4) = 4 \\
= \frac{11}{4} \left( \frac{1}{1} \right) = 0 \\
= 0 - \frac{11}{4}(0) = 0 \\
\]

The above information is summarized in the following table.

**Simplex Table III**

<table>
<thead>
<tr>
<th>( c_j \rightarrow )</th>
<th>Solution</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_g )</td>
<td>Basis</td>
<td>( b = x_{\text{old}} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( x_2 )</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>( \frac{1}{3} )</td>
<td>0</td>
<td>( \frac{4}{3} )</td>
</tr>
<tr>
<td>0</td>
<td>( s_2 )</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>( \frac{2}{3} )</td>
<td>1</td>
<td>( \frac{11}{3} )</td>
</tr>
<tr>
<td>2</td>
<td>( x_1 )</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>( -\frac{1}{3} )</td>
<td>0</td>
<td>( \frac{4}{3} )</td>
</tr>
<tr>
<td>2 = 33</td>
<td>( Z ) = ( c_j - c_{b_j} )</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\* \( Z = \Sigma c_j x_j = 5(5) + 0(4) + 2(4) = 33 \)

As all entries in \( C_j \)-row are either negative or zero, the above table gives the optimal basic feasible solution.

\* The optimal solution is \( x_1 = 4 \), \( x_2 = 5 \) and \( Max. Z = 33 \).

**Example 2.** Use the simplex method to solve the following LP problem.

Maximize \( Z = 3x_1 + 5x_2 + 4x_3 \)

subject to
\[
2x_1 + 3x_2 + s_1 = 8 \\
2x_2 + 5x_3 + s_2 = 10 \\
3x_1 + 2x_2 + 4x_3 + s_3 = 15 \\
x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\]

**Sol. Step 1.** Introducing slack variables \( s_1, s_2, s_3 \) (one for each constraint) the problem in standard form is:

Maximize \( Z = 3x_1 + 5x_2 + 4x_3 + s_1 + 0s_2 + 0s_3 \)

subject to
\[
2x_1 + 3x_2 + s_1 = 8 \quad \ldots (1) \\
2x_2 + 5x_3 + s_2 = 10 \quad \ldots (2) \\
3x_1 + 2x_2 + 4x_3 + s_3 = 15 \quad \ldots (3) \\
x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\]

Step 2. Since we have 3 equations in 6 variables, a solution is obtained by setting \( 6 - 3 = 3 \) variables equal to zero and solving for the remaining 3 variables. We choose initial basic feasible solution as:
\[
x_1 = x_2 = x_3 = 0; \quad s_1 = 8, \quad s_2 = 10, \quad s_3 = 15
\]

at which \( Z = 0 \).
Simplex Table I

<table>
<thead>
<tr>
<th>$c_j$</th>
<th>Basis</th>
<th>Solution</th>
<th>Basis</th>
<th>Solution</th>
<th>Basis</th>
<th>Solution</th>
<th>Basis</th>
<th>Solution</th>
<th>Basis</th>
<th>Solution</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_j \rightarrow )</td>
<td>( \text{b} = \text{x}_b )</td>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>( x_3 )</td>
<td>( x_4 )</td>
<td>( x_5 )</td>
<td>( x_6 )</td>
<td>( x_7 )</td>
<td>( x_8 )</td>
<td>( x_9 )</td>
<td>( x_{10} )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( z_1 )</td>
<td>8</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( z_2 )</td>
<td>10</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( z_3 )</td>
<td>15</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ Z = 0 \]

\[ Z_j = c_j - Z_j \]

Step 3. Since some entries in $C_j$ row are positive, the current solution is not optimal.

The incoming variable is $x_3$, and the outgoing basic variable is $z_1$. Also 3 is the key number.

Step 4. The co-efficient of $x_3$ in the objective function is 5. Therefore the entry in $c_j$ column corresponding to the new basic variable $x_3$ will be 5. Dividing all elements of the key row by the key element 3, the **new key row** is

\[ \begin{align*}
5 & \quad 2 & \quad 1 & \quad 0 & \quad 1 & \quad 0 & \quad 0 \\
3 & \quad 3 & \quad 3 & \quad 3 & \quad 3 & \quad 3 & \quad 3
\end{align*} \]

Transformation of $R_2$. Key column entry in $R_2$ is 2.

\[ \begin{align*}
R_2(\text{new}) &= R_2(\text{old}) - 2R_1(\text{new}) \\
10 & \quad - \frac{8}{3} \quad \frac{14}{3} \\
0 & \quad - \frac{2}{3} \quad \frac{4}{3} \\
2 & \quad 2(1) = 0 \\
5 & \quad 2(0) = 5 \\
0 & \quad - \frac{1}{3} \quad \frac{2}{3} \\
1 & \quad 2(0) = 1 \\
0 & \quad - \frac{2}{3} \quad \frac{0}{3}
\end{align*} \]

Transformation of $R_3$. Key column entry in $R_3$ is 2.

\[ \begin{align*}
R_3(\text{new}) &= R_3(\text{old}) - 2R_1(\text{new}) \\
15 & \quad - \frac{8}{3} \quad \frac{29}{3} \\
3 & \quad - \frac{2}{3} \quad \frac{5}{3} \\
3 & \quad - \frac{2}{3} \quad \frac{5}{3}
\end{align*} \]

The above information is summarized in the following table.

Simplex Table II

<table>
<thead>
<tr>
<th>$c_j$</th>
<th>Basis</th>
<th>Solution</th>
<th>Basis</th>
<th>Solution</th>
<th>Basis</th>
<th>Solution</th>
<th>Basis</th>
<th>Solution</th>
<th>Basis</th>
<th>Solution</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_j \rightarrow )</td>
<td>( \text{b} = \text{x}_b )</td>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>( x_3 )</td>
<td>( x_4 )</td>
<td>( x_5 )</td>
<td>( x_6 )</td>
<td>( x_7 )</td>
<td>( x_8 )</td>
<td>( x_9 )</td>
<td>( x_{10} )</td>
</tr>
<tr>
<td>( 5 )</td>
<td>( z_2 )</td>
<td>8</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( z_3 )</td>
<td>14</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( z_3 )</td>
<td>29</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ Z = 3 \]

\[ Z_j = c_j - Z_j \]

\[ \begin{align*}
\ast & \quad Z = 5 \left( \frac{8}{3} \right) + 0 \left( \frac{14}{3} \right) + 0 \left( \frac{29}{3} \right) = \frac{40}{3} \\
\end{align*} \]

Since all entries in $C_j$ row are not negative or zero, the current solution is not optimal.

The incoming variable is $x_3$, and the outgoing basic variable is $z_3$. Also 5 is the key number.

Step 5. The co-efficient of $x_3$ in the objective function is 4. Therefore the entry in $c_j$ column corresponding to the new basic variable $x_3$ will be 4. Dividing all elements of the key row by the key element 5, the **new key row** is

\[ \begin{align*}
4 & \quad - \frac{8}{3} \quad \frac{14}{3} \quad \frac{29}{3} \\
0 & \quad - \frac{2}{3} \quad \frac{4}{3} \quad \frac{0}{3} \\
2 & \quad 2(1) = 0 \\
0 & \quad 2(0) = 5 \\
0 & \quad - \frac{1}{3} \quad \frac{2}{3} \\
1 & \quad 2(0) = 1 \\
0 & \quad - \frac{2}{3} \quad \frac{0}{3}
\end{align*} \]

Transformation of $R_2$. Key column entry in $R_2$ is 0.

\[ R_2(\text{new}) = R_2(\text{old}) - 0 \times R_1(\text{new}) = R_2(\text{old}) \]

Transformation of $R_3$. Key column entry in $R_3$ is 4.

\[ R_3(\text{new}) = R_3(\text{old}) - 4R_1(\text{new}) \]

\[ \begin{align*}
\frac{29}{3} & \quad - \frac{4}{3} \quad \frac{82}{15} \\
\frac{5}{3} & \quad - \frac{4}{3} \quad \frac{41}{15}
\end{align*} \]
The above information is summarized in the following table.

### Simplex Table III

<table>
<thead>
<tr>
<th>( c_i )</th>
<th>Basis</th>
<th>( c_i \rightarrow ) Solution</th>
<th>3</th>
<th>5</th>
<th>4</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>( \text{Ratio} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( x_2 )</td>
<td>( \frac{8}{3} )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{(8)}{(3)} = \frac{8}{3} )</td>
</tr>
<tr>
<td>4</td>
<td>( x_3 )</td>
<td>( \frac{14}{15} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{(14)}{(15)} = \frac{14}{15} )</td>
</tr>
<tr>
<td>0</td>
<td>( s_3 )</td>
<td>( \frac{89}{15} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{(89)}{(15)} = \frac{89}{15} )</td>
</tr>
<tr>
<td><strong>Z</strong></td>
<td>( \frac{256}{15} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{256}{15} )</td>
</tr>
</tbody>
</table>

\( Z = \frac{5}{3} \left( \frac{8}{3} + 4 \frac{14}{15} + 0 \frac{89}{15} \right) = \frac{256}{15} \)

Since all entries in \( C_r \) row are not negative or zero, the current solution is not optimal.

The incoming variable is \( x_1 \) and the outgoing basic variable is \( s_3 \). Also \( \frac{41}{15} \) is the key number.

**Step 6.** The co-efficient of \( x_1 \) in the objective function is 3. Therefore the entry in \( c_3 \) column corresponding to the new basic variable \( x_1 \) will be 3. Dividing all elements of the key row by the key element \( \frac{41}{15} \) the **new key row** is

\[
\begin{align*}
3 & x_1 \\
\frac{89}{41} & 1 \quad 0 \quad 0 \quad 0 \quad \frac{2}{41} \quad \frac{12}{41} \quad \frac{15}{41}
\end{align*}
\]

### Transformation of \( R_1 \)

Key column entry in \( R_1 \) is \( \frac{2}{3} \)

\[
R_1(\text{new}) = R_1(\text{old}) - \frac{2}{3} R_2(\text{new})
\]

\[
\frac{8}{3} - 2 \frac{50}{41} = \frac{50}{41}
\]

\[
\frac{3}{3} - 3 \frac{89}{15} = \frac{89}{15}
\]

The above information is summarized in the following table.

### Simplex Table IV

<table>
<thead>
<tr>
<th>( c_i )</th>
<th>Basis</th>
<th>( c_i \rightarrow ) Solution</th>
<th>3</th>
<th>5</th>
<th>4</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>( \text{Ratio} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( x_2 )</td>
<td>50</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{50}{41} = \frac{50}{41} )</td>
</tr>
<tr>
<td>4</td>
<td>( x_3 )</td>
<td>41</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{41}{41} = \frac{41}{41} )</td>
</tr>
<tr>
<td>3</td>
<td>( x_1 )</td>
<td>41</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{41}{41} = \frac{41}{41} )</td>
</tr>
<tr>
<td><strong>Z</strong></td>
<td>( \frac{765}{41} )</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{765}{41} )</td>
</tr>
</tbody>
</table>

\( Z = \frac{c_j - c_i}{C_r} = \frac{50}{41} \)
Since all entries in \( C_2 \)-row are either negative or zero, therefore, the optimal solution is reached with

\[
x_1 = \frac{50}{41}, \quad x_2 = \frac{41}{41}, \quad x_3 = \frac{62}{41}, \quad \text{and Max. } Z = 765
\]

**Example 3.** Solve the following linear programming problem by simplex method:

Maximize \( Z = 2x_1 + x_2 - x_3 \)

subject to

\[
\begin{align*}
x_1 + x_2 & \leq 1 \\
x_2 + 2x_3 & \leq 2 \\
x_1, x_2, x_3 & \geq 0.
\end{align*}
\]

**Sol. Step 1.** The problem is of maximization. As the sign of \( b \) in the second inequality is negative, it is first converted to positive by multiplying both the sides of the second inequality by (-1). Thus, the problem can be restated as

Maximize \( Z = 2x_1 + x_2 - x_3 \)

subject to

\[
\begin{align*}
x_1 + x_2 & \leq 1 \\
x_2 + 2x_3 & \geq 2 \\
x_1, x_2, x_3 & \geq 0.
\end{align*}
\]

**Step 2.** Introducing slack variables \( s_1, s_2 \) (one for each constraint) the problem in standard form is:

Maximize \( Z = 2x_1 + x_2 - x_3 + 0s_1 + 0s_2 \)

subject to

\[
\begin{align*}
x_1 + x_2 + s_1 & = 1 \\
x_2 + 2x_3 + s_2 & = 2 \\
x_1, x_2, x_3, s_1, s_2 & \geq 0.
\end{align*}
\]

**Step 3.** Since we have 2 equations in 5 variables, a solution is obtained by setting 3 - 2 = 1 variables equal to zero and solving for the remaining 2 variables. We choose initial basic feasible solution as:

\[
\begin{align*}
x_1 = x_2 = x_3 = 0; \quad s_1 = 1, \quad s_2 = 2
\end{align*}
\]

at which \( Z = 0 \)

**Simplex Table I**

<table>
<thead>
<tr>
<th>( c_i )</th>
<th>Basis</th>
<th>( c_i \rightarrow ) Solution ( b(=x_0) )</th>
<th>2</th>
<th>1</th>
<th>-1</th>
<th>0</th>
<th>0</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_0 )</td>
<td>( s_1 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \frac{1}{2} = 1 \rightarrow )</td>
<td></td>
</tr>
<tr>
<td>( c_0 )</td>
<td>( s_2 )</td>
<td>2</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( Z = 0 )</td>
<td>( C_1 = c_j - Z )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Step 4.** Since some entries in \( C_j \)-row are positive, the current solution is not optimal. The incoming variable is \( x_1 \) and the outgoing basic variable is \( s_1 \). Also 1 is the key number.

**Linear Programming**

Since the key number is 1, new key row is the same as old one.

**Transformation of \( R_1 \).** Key column entry in \( R_2 \) is \((-1)\)

\[
R_{new} = R_{old} + 1 \cdot R_{new}
\]

\[
\begin{align*}
2 + 1(1) &= 3 \\
-1 + 1(1) &= 0 \\
2 + 1(1) &= 3 \\
1 + 1((-1)) &= 1 \\
0 + 1(1) &= 1 \\
1 + 1((-1)) &= 1
\end{align*}
\]

The above information is summarized in the following table.

**Simplex Table II**

<table>
<thead>
<tr>
<th>( c_i )</th>
<th>Basis</th>
<th>( c_i \rightarrow ) Solution ( b(=x_0) )</th>
<th>2</th>
<th>1</th>
<th>-1</th>
<th>0</th>
<th>0</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_0 )</td>
<td>( x_1 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( c_0 )</td>
<td>( s_2 )</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( Z = 2 )</td>
<td>( C_1 = c_j - Z )</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Since all entries in \( C_j \)-row are either negative or zero, therefore, the optimal solution is reached with

\[
\begin{align*}
x_1 &= 1; \quad x_2 = 0; \quad x_3 = 0 \quad \text{and Max. } Z = 2
\end{align*}
\]

**Example 4.** Solve the following LPP by simplex method.

Minimize \( Z = x_1 - 3x_2 + 3x_3 \)

subject to

\[
\begin{align*}
3x_1 - x_2 + 2x_3 & \leq 7 \\
2x_1 + 4x_2 & \geq 12 \\
-4x_1 + 3x_2 + 8x_3 & \leq 10 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

(M.D.U. May 2011)

**Sol. Step 1.** The problem is that of minimization. Converting it into maximization problem by using the relationship

\[
\text{Min. } Z = - \text{Max. } Z^* \quad \text{where } Z^* = -Z
\]

As the sign of \( b \) in the second inequality is negative, it is first converted to positive by multiplying both the sides of the second inequality by \((-1)\). Thus, the problem can be restated as

Maximize \( Z^* = -x_1 + 3x_2 - 3x_3 \)

subject to

\[
\begin{align*}
3x_1 - x_2 + 2x_3 & \leq 7 \\
-2x_1 - 4x_2 & \leq -12 \\
-4x_1 + 3x_2 + 8x_3 & \leq 10 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]
Step 2. Introducing slack variables $s_1, s_2, s_3$ (one for each constraint), the problem in standard form is:
Maximize $Z^* = -x_1 + 3x_2 - 3x_3 + 0x_1 + 0s_2 + 0s_3 \\
subject to \\
\begin{align*}
3x_1 - x_2 + 2x_3 + s_1 &= 7 \\
-2x_1 - 4x_2 + s_2 &= 12 \\
-4x_1 + 3x_2 + 8x_3 + s_3 &= 10
\end{align*} \\
x_1, x_2, x_3, s_1, s_2, s_3 \geq 0

Step 3. Since we have 3 equations in 6 variables, a solution is obtained by setting $6 - 3 = 3$ variables equal to zero and solving for the remaining 3 variables. We choose initial basic feasible solution as:
$x_1 = x_2 = x_3 = 0; s_1 = 7, s_2 = 12, s_3 = 10$ at which $Z^* = 0$

**Simplex Table I**

<table>
<thead>
<tr>
<th>$c_p$</th>
<th>Basis</th>
<th>$c_j \rightarrow$ Solution</th>
<th>$b = x_p$</th>
<th>$-1$</th>
<th>3</th>
<th>-3</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x_1$</td>
<td>7</td>
<td>3</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$x_2$</td>
<td>12</td>
<td>-2</td>
<td>3</td>
<td>8</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{10}{3}$</td>
</tr>
<tr>
<td>0</td>
<td>$x_3$</td>
<td>10</td>
<td>-4</td>
<td>3</td>
<td>8</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{10}{3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Z^*_p$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

**Step 4.** Since some entries in $C_j$ row are positive, the current solution is not optimal. The incoming variable is $x_2$ and the outgoing basic variable is $s_2$. Also the key number is 3.

Dividing all elements of the key row by the key number 3, the **new key row** is

\[
\begin{array}{ccccccc}
3 & x_2 & \frac{10}{3} & \frac{4}{3} & 1 & \frac{8}{3} & 0 & 0 & \frac{1}{3}
\end{array}
\]

**Transformation of $R_1$, Key column entry in $R_2$ is (-1)**

\[
R_1 (\text{new}) = R_1 (\text{old}) + \frac{R_1}{\text{new}} (\text{new})
\]

\[
\begin{array}{ccccccc}
7 + \frac{10}{3} &=& \frac{31}{3} \\
3 + \frac{4}{3} &=& \frac{5}{3} \\
-1 + 1(1) &=& 0 \\
2 + \frac{8}{3} &=& \frac{14}{3}
\end{array}
\]

**Linear Programming**

\[
\begin{array}{c}
1 + 1(0) = 1 \\
0 + 1(0) = 0 \\
0 + \left(1\right) = \frac{1}{3}
\end{array}
\]

**Transformation of $R_2$, Key column entry in $R_3$ is (-4)**

\[
R_2 (\text{new}) = R_2 (\text{old}) + \frac{R_2}{\text{new}} (\text{new})
\]

\[
\begin{array}{ccccccc}
12 + 4 \left(\frac{10}{3}\right) &=& \frac{76}{3} \\
-2 + 4 \left(-\frac{4}{3}\right) &=& \frac{-22}{3} \\
-4 + 4(1) &=& 0 \\
0 + 4(0) &=& 0 \\
1 + 4(0) &=& 1 \\
0 + 4 \left(\frac{1}{3}\right) &=& \frac{4}{3}
\end{array}
\]

The above information is summarized in the following table.

**Simplex Table II**

<table>
<thead>
<tr>
<th>$c_p$</th>
<th>Basis</th>
<th>$c_j \rightarrow$ Solution</th>
<th>$b = x_p$</th>
<th>$-1$</th>
<th>3</th>
<th>-3</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>Ratio $x_j/x_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$s_1$</td>
<td>$\frac{31}{3}$</td>
<td>$\frac{5}{3}$</td>
<td>0</td>
<td>14</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\frac{31}{5}$</td>
</tr>
<tr>
<td>0</td>
<td>$s_2$</td>
<td>76</td>
<td>$\frac{-22}{3}$</td>
<td>0</td>
<td>32</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>$\frac{-22}{3}$</td>
</tr>
<tr>
<td>3</td>
<td>$x_2$</td>
<td>10</td>
<td>$\frac{-4}{3}$</td>
<td>3</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>$\frac{-4}{3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Z^*_p$</td>
<td>$\frac{-4}{3}$</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>$\frac{-4}{3}$</td>
</tr>
</tbody>
</table>

**Step 5.** Since all entries in $C_j$ row are not negative or zero, the current solution is not optimal.

The incoming variable is $x_1$, and the outgoing basic variable is $s_1$. Also the key number is $\frac{5}{3}$.
Dividing all elements of the key row by the key number $\frac{5}{3}$, the new key row is

$$-1 \begin{array}{c} x_1 \\ \frac{31}{5} \\ 1 \\ 0 \\ \frac{14}{5} \\ \frac{3}{5} \\ 0 \end{array} \begin{array}{c} \end{array} \begin{array}{c} \frac{1}{5} \end{array}$$

Transformation of $R_2$. Key column entry in $R_2$ is $- \frac{22}{3}$.

$$R_2(\text{new}) = R_2(\text{old}) + \frac{22}{3} R_1(\text{new})$$

$$\begin{array}{c} \frac{76}{3} \cdot \frac{22}{3} \cdot \frac{31}{5} \\ \frac{-22}{3} \cdot \frac{22}{3} \cdot \frac{1}{1} = 0 \\ \frac{0}{3} \cdot \frac{22}{3} \cdot \frac{0}{1} = 0 \\ \frac{32}{3} \cdot \frac{22}{3} \cdot \frac{14}{5} = \frac{156}{5} \\ \frac{0}{3} \cdot \frac{22}{3} \cdot \frac{3}{5} = \frac{22}{5} \\ \frac{1}{3} \cdot \frac{22}{3} \cdot \frac{0}{1} = 1 \\ \frac{4}{3} \cdot \frac{22}{3} \cdot \frac{1}{5} = \frac{14}{5} \end{array}$$

Transformation of $R_3$. Key column entry in $R_3$ is $- \frac{4}{3}$.

$$R_3(\text{new}) = R_3(\text{old}) + \frac{4}{3} R_1(\text{new})$$

$$\begin{array}{c} \frac{10}{3} \cdot \frac{4}{3} \cdot \frac{31}{5} = \frac{58}{5} \\ \frac{-4}{3} \cdot \frac{4}{3} \cdot \frac{1}{1} = 0 \\ \frac{1}{3} \cdot \frac{4}{3} \cdot \frac{0}{1} = 1 \\ \frac{8}{3} \cdot \frac{4}{3} \cdot \frac{14}{5} = \frac{32}{5} \\ \frac{0}{3} \cdot \frac{4}{3} \cdot \frac{3}{5} = \frac{4}{5} \end{array}$$

The above information is summarized in the following table.

<table>
<thead>
<tr>
<th>Simplex Table III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_i$</td>
</tr>
<tr>
<td>-------------------</td>
</tr>
<tr>
<td>$x_1$</td>
</tr>
<tr>
<td>$x_2$</td>
</tr>
<tr>
<td>$x_3$</td>
</tr>
<tr>
<td>$Z^* = \frac{143}{5}$</td>
</tr>
<tr>
<td>$c_j = c_j - Z^*_j$</td>
</tr>
</tbody>
</table>

Since all entries in $C_j$ row are either negative or zero, therefore, the optimal solution is reached with

$$x_1 = \frac{31}{5}, x_2 = \frac{58}{5}, x_3 = 0$$

and Max. $Z^* = \frac{143}{5}$.

Hence, from (1), Min. $Z = -\frac{143}{5}$.

Example 5. Following data are available for a farm which manufactures three items A, B and C:

<table>
<thead>
<tr>
<th>Product</th>
<th>Time required in hours</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Assembly</td>
</tr>
<tr>
<td>---------</td>
<td>-----------</td>
</tr>
<tr>
<td>A</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
</tr>
<tr>
<td>Firm's Capacity</td>
<td>2000</td>
</tr>
</tbody>
</table>

Express the above data in the form of linear programming problem to maximize the profit from the production and solve it by simplex method.
Sol. Step 1. Let \( x_1, x_2, x_3 \) denote respectively the number of units of products A, B, C manufactured by the firm.

The mathematical formulation of the problem is:

Maximize \( Z = 80x_1 + 60x_2 + 30x_3 \)

subject to the constraints

\[
\begin{align*}
10x_1 + 4x_2 + 5x_3 & \leq 2000 \\
2x_1 + 5x_2 + 4x_3 & \leq 1009 \\
x_1, x_2, x_3, s_1, s_2 & \geq 0
\end{align*}
\]

Step 2. Introducing slack variables \( s_1, s_2 \) (one for each constraint) the problem in standard form is:

Maximize \( Z = 80x_1 + 60x_2 + 30x_3 + 0s_1 + 0s_2 \)

subject to

\[
\begin{align*}
10x_1 + 4x_2 + 5x_3 + s_1 & = 2000 \\
2x_1 + 5x_2 + 4x_3 + s_2 & = 1009 \\
x_1, x_2, x_3, s_1, s_2 & \geq 0
\end{align*}
\]

Step 3. Since we have 2 equations in 3 variables, a solution is obtained by setting 5 - 2 = 3 variables equal to zero and solving for the remaining 2 variables.

An initial basic feasible solution is obtained by setting \( x_1 = x_2 = x_3 = 0 \) so that \( s_1 = 2000, s_2 = 1009 \).

The initial simplex table corresponding to this solution is given below:

**Simplex Table I**

| \( c_8 \) | Basis | \( c_i \rightarrow \) Solution \( b = x_p \) | \( 80 \) | \( 60 \) | \( 30 \) | \( 0 \) | \( 0 \) | Ratio |
|---|---|---|---|---|---|---|---|
| 0 | \( s_1 \) | 2000 | 0 | 0 | 0 | 0 | 0 | 200 |
| 0 | \( s_2 \) | 1009 | 10 | 4 | 5 | 1 | 0 | 1009 |
| \( Z = 0 \) | \( Z_j \) | \( c_j = c_j - Z_j \) | 80 | 60 | 80 | 0 | 0 | 2 |

Step 4. Since \( c_j \)-row has some positive entries, the current solution is not optimal.

The incoming variable is \( x_1 \) and the outgoing basic variable is \( s_1 \). The key number is 10.

Dividing each element in the key row \( i.e., \ s_1 \)-row, by the key number 10 the new key row is

\[
\begin{align*}
80 & \quad x_1 \quad 200 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \\
5 & \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0
\end{align*}
\]

Transformation of \( R_2 \). Key column entry in \( R_2 \) is 2.

\[
\frac{R_2(\text{new}) = R_2(\text{old}) - \frac{2}{5} R_3(\text{new})}{1009 - 2(200) = 609}
\]

\[
\frac{2 - (21)}{0} = 9
\]

The above information is summarized in the following table.

**Simplex Table II**

| \( c_8 \) | Basis | \( c_i \rightarrow \) Solution \( b = x_p \) | \( 80 \) | \( 60 \) | \( 30 \) | \( 0 \) | \( 0 \) | Ratio |
|---|---|---|---|---|---|---|---|
| 80 | \( x_1 \) | 200 | 1 | 2 | 1 | 1 | 0 |
| 0 | \( s_2 \) | 609 | 0 | 0 | 0 | 0 | 0 |
| \( Z = 16000 \) | \( Z_j \) | \( c_j = c_j - Z_j \) | 80 | 60 | 10 | 0 | 0 | 2 |

Step 5. Since \( C_j \)-row has a positive entry, the current solution is not optimal.

The incoming variable is \( x_2 \) and the outgoing basic variable is \( s_2 \). The key number is \( \frac{21}{5} \).

Dividing each element in the key row by the key number \( \frac{21}{5} \), the new key row is

\[
\begin{align*}
80 & \quad x_2 \quad 145 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \\
5 & \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0
\end{align*}
\]

**Transformation of \( R_1 \).** Key column entry in \( R_2 \) is \( \frac{2}{5} \).

\[
\frac{R_1(\text{new}) = R_1(\text{old}) - \frac{2}{5} R_2(\text{new})}{200 - \frac{2}{5} (142) = 142}
\]

\[
\frac{2 - (0)}{0} = 0
\]

\[
\frac{1}{2} \quad \frac{2}{5} \quad \frac{1}{7} \quad \frac{1}{21} \quad \frac{5}{21}
\]

\[
\frac{1}{2} \quad \frac{2}{5} \quad \frac{1}{7} \quad \frac{1}{21} \quad \frac{5}{21}
\]
that we may never obtain the optimal value of the objective function. However, this phenomenon seldom occurs in practical problems.

Degeneracy may occur at any iteration of the simplex method. When there is a tie in the iterations required to arrive at the optimal solution can be minimized by adopting the follow-

(i) Identify the tied rows.

(ii) Divide the co-efficients of slack variables in each tied row by the corresponding posi-

(iii) Compare the resulting ratios columnwise from left to right.

(iv) Select the row which has the smallest ratio. This row becomes the key row.

If the above ratios fail to break the tie, then find similar ratios for the decision vari-

Thus, the degeneracy is resolved. Simplex method is then continued until an optimal solution is obtained.

**Example. Using simplex method**

Maximize  \( Z = 5x_1 + 3x_2 \)

subject to

\[
\begin{align*}
7x_1 + 2x_2 & \leq 20 \\
3x_1 + 8x_2 & \leq 12 \\
x_1, x_2 & \geq 0
\end{align*}
\]

**Sol. Step 1.** Introducing slack variables \( s_1, s_2, s_3 \) (one for each constraint) the problem in standard form is:

Maximize  \( Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3 \)

subject to

\[
\begin{align*}
7x_1 + 2x_2 + s_1 & = 20 \\
3x_1 + 8x_2 + s_2 & = 12 \\
x_1, x_2, s_1, s_2 & \geq 0
\end{align*}
\]

**Step 2.** Since we have 3 equations in 5 variables, a solution is obtained by setting 5 \(-3=2\) variables equal to zero and solving for the remaining 3 variables.

Thus an initial basic feasible solution is obtained by setting \( x_1 = x_2 = 0 \) so that \( s_1 = 2, s_2 = 10, s_3 = 12 \).

The initial simplex table corresponding to this solution is given below:

**Simplex Table I**

<table>
<thead>
<tr>
<th>( c_B )</th>
<th>Basis</th>
<th>( c_j \rightarrow ) Solution ( b(=x_B) )</th>
<th>( 5 )</th>
<th>( 3 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( s_1 )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( \frac{5}{2} ) Tied</td>
</tr>
<tr>
<td>0</td>
<td>( s_2 )</td>
<td>10</td>
<td>(5)</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( s_3 )</td>
<td>12</td>
<td>3</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>( Z_j )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( C_j = c_j - Z_j )</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The above information is summarized in the following table.

**Simplex Table III**

<table>
<thead>
<tr>
<th>( c_B )</th>
<th>Basis</th>
<th>( c_j \rightarrow ) Solution ( b(=x_B) )</th>
<th>( 80 )</th>
<th>( 60 )</th>
<th>( 30 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>( x_1 )</td>
<td>142</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>( x_2 )</td>
<td>145</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>21</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>( x_1 )</td>
<td>142</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>( x_2 )</td>
<td>145</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>21</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Z = 20060</td>
<td>( Z_j )</td>
<td>80</td>
<td>80</td>
<td>60</td>
<td>60</td>
<td>20</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Z = 20060</td>
<td>( C_j = c_j - Z_j )</td>
<td>0</td>
<td>0</td>
<td>-30</td>
<td>3</td>
<td>-20</td>
<td>-20</td>
<td></td>
</tr>
</tbody>
</table>

Since all entries in \( C_j \)-row are either negative or zero, the optimal solution has been obtained with \( x_1 = 142, x_2 = 145 \) and Max. \( Z = 20060 \).

7.11. TIE FOR THE INCOMING VARIABLE (Key Column)

At any iteration, if two or more columns have the same largest positive value in \( C_j \)-row, a tie for the incoming variable occurs. In order to break this tie, the selection for key column (incoming variable) can be made arbitrarily. However, in order to minimize the number of iterations required to reach the optimal solution, the following rules may be used:

(i) If there is a tie between two decision variables, the choice can be made arbitrarily.

(ii) If there is a tie between two slack (or surplus) variables, the choice can be made arbitrarily.

(iii) If there is a tie between a decision variable and a slack (or surplus) variable, then the decision variable is selected as the incoming variable.

7.12. TIE FOR THE OUTGOING VARIABLE (Key Row)—DEGENERACY

In the simplex method, in order to determine which basic variable to go out of the basis (i.e., to determine the key row), we find the ratio of entries in \( x_B \)-column by the corresponding (but positive) entries in the key column and select that variable to go out of the basis for which this ratio is the minimum. Sometimes this minimum ratio is not unique or values of one or more basic variables in the \( x_B \)-column are zero. This problem is known as degeneracy in linear programming and the basic feasible solution is degenerate. It is possible, in a degenerate situation, to obtain a sequence of simplex tables that correspond to basic feasible solutions which give the same value of the objective function. Moreover, we may eventually return to the first simplex table in the sequence. This is called cycling. When cycling occurs, it is possible...
Since \( C_j \)-row has some positive entries, the current solution is not optimal. The incoming variable is \( x_1 \).

**Step 3.** From Table I we observe that there is a tie among the \( s_1 \)-row and \( s_2 \)-row. This is an indication of the existence of degeneracy.

To obtain the unique key row, we adopt the following procedure:

(i) The co-efficients of slack variables \( s_1 \) and \( s_2 \) are:

<table>
<thead>
<tr>
<th>Row</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(ii) Key column element in \( s_1 \)-row is 1 and in \( s_2 \)-row is 5.

Dividing the co-efficients in \( s_1 \)-row by 1 and in \( s_2 \)-row by 5, we get the following ratios:

<table>
<thead>
<tr>
<th>Row</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

(iii) Comparing the ratios of Step (ii) from left to right columnwise, we find that in the first column, the second row yields the smaller ratio and, therefore, \( s_2 \)-row becomes the key row and \( s_1 \) is the outgoing basic variable. The key number is 5.

**Step 4.** Dividing each element in the key row by the key number 5, the new key row is

\[
\begin{align*}
5 & x_1 & 2 & 1 & 2/5 & 0 & 1/5 & 0 \\
\end{align*}
\]

**Transformation of \( R_1 \).** Key column entry in \( R_1 \) is 1

\[
\begin{align*}
R_1(\text{new}) = R_1(\text{old}) - 1 \cdot R_2(\text{new}) \\
2 - 1(2) & = 0 \\
1 - 1(1) & = 0 \\
1 - 1(2/5) & = 3/5 \\
1 - 1(0) & = 1 \\
0 - 1(1/5) & = -1/5 \\
0 - 1(0) & = 0 \\
\end{align*}
\]

**Transformation of \( R_2 \).** Key column entry in \( R_2 \) is 3

\[
\begin{align*}
R_2(\text{new}) = R_2(\text{old}) - 3 \cdot R_2(\text{new}) \\
12 - 3(2) & = 6 \\
3 - 3(1) & = 0 \\
\end{align*}
\]

The above information is summarized in the following table.

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>Basis</th>
<th>Solution b(= ( x^p ))</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( x_p/x_2 )</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( s_1 )</td>
<td>0</td>
<td>0</td>
<td>3/5</td>
<td>1</td>
<td>-1/5</td>
<td>0</td>
<td>0</td>
<td>( \infty )</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>( x_1 )</td>
<td>2</td>
<td>1</td>
<td>2/5</td>
<td>0</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( s_3 )</td>
<td>6</td>
<td>34</td>
<td>5/5</td>
<td>0</td>
<td>-3/5</td>
<td>1</td>
<td>15/17</td>
<td>17</td>
<td></td>
</tr>
</tbody>
</table>

\[ Z = 10 \]

\[ C_j = c_j - Z_j \]

Since \( C_j \)-row has a positive entry, the current solution is not optimal. The incoming variable is \( x_2 \) and the outgoing basic variable is \( s_1 \). The key number is \( \frac{3}{5} \).

Dividing each element in the key row by the key number \( \frac{3}{5} \), the new key row is

\[
\begin{align*}
3 & x_2 & 0 & 0 & 1 & 5/3 & 1/3 & 0 \\
\end{align*}
\]

**Transformation of \( R_3 \).** Key column entry in \( R_3 \) is \( \frac{2}{5} \)

\[
\begin{align*}
R_3(\text{new}) = R_3(\text{old}) - \frac{2}{5} \cdot R_2(\text{new}) \\
2 - 2(0) & = 2 \\
1 - 2(0) & = 1 \\
2 - 2(1) & = 0 \\
0 - 2(5/3) & = 2 \\
0 - 2(5/3) & = 3 \\
\end{align*}
\]
Using simplex method, solve the following linear programming problems (1-6):

1. (a) Maximize \( Z = x_1 + 2x_2 \)
   subject to \( \begin{align*}
   2x_1 + x_2 & \leq 8 \\
   5x_1 + 2x_2 & \leq 12 \\
   x_1, x_2 & \geq 0
   \end{align*} \) (M.D.U. Dec. 2013)

   (ii) Maximize \( Z = x_1 + 3x_2 \)
   subject to \( \begin{align*}
   x_1 + 2x_2 & \leq 10 \\
   0 \leq x_1 \leq 5 \\
   0 \leq x_2 \leq 4
   \end{align*} \)

   (iii) Maximize \( Z = 6x_1 + 4x_2 \)
   subject to \( \begin{align*}
   x_1 + 2x_2 & \leq 20 \\
   3x_1 + 2x_2 & \leq 30 \\
   x_1, x_2 & \geq 0
   \end{align*} \)

   (iv) Maximize \( Z = 4x_1 + 5x_2 \)
   subject to \( \begin{align*}
   x_1 - 2x_2 & \geq 2 \\
   2x_1 + x_2 & \leq 6 \\
   x_1 + 2x_2 & \leq 5 \\
   -x_1 + x_2 & \leq 2 \\
   x_1, x_2 & \geq 0
   \end{align*} \)

   (v) Maximize \( Z = 45x_1 + 80x_2 \)
   subject to \( \begin{align*}
   5x_1 + 20x_2 & \leq 400 \\
   10x_1 + 15x_2 & \leq 450 \\
   x_1, x_2 & \geq 0
   \end{align*} \) (M.D.U. May 2009)

2. (a) Maximize \( Z = 10x_1 + x_2 + 2x_3 \)
   subject to \( \begin{align*}
   x_1 + x_2 - 2x_3 & \leq 10 \\
   4x_1 + 2x_2 + x_3 & \leq 20 \\
   x_1, x_2, x_3 & \geq 0
   \end{align*} \)

   (ii) Maximize \( Z = 5x_1 + 4x_2 + 3x_3 \)
   subject to \( \begin{align*}
   3x_1 + 2x_2 + x_3 & \leq 10 \\
   2x_1 + x_2 + 2x_3 & \leq 12 \\
   x_1 + 2x_2 + 3x_3 & \leq 15 \\
   x_1, x_2, x_3 & \geq 0
   \end{align*} \)

   (iii) Maximize \( Z = x_1 + x_2 + 3x_3 \)
   subject to \( \begin{align*}
   3x_1 + 2x_2 + x_3 & \leq 3 \\
   2x_1 + x_2 + 2x_3 & \leq 2 \\
   x_1, x_2, x_3 & \geq 0
   \end{align*} \)

   (iv) Maximize \( Z = 4x_1 + 3x_2 + 4x_3 + 6x_4 \)
   subject to \( \begin{align*}
   x_1 + 2x_2 + 2x_3 + 4x_4 & \leq 80 \\
   2x_1 + 2x_2 + x_3 & \leq 60 \\
   3x_1 + 3x_2 + x_4 & \leq 80 \\
   x_1, x_2, x_3, x_4 & \geq 0
   \end{align*} \)

3. (a) Maximize \( Z = x_1 + 2x_2 + x_3 \)
   subject to \( \begin{align*}
   2x_1 + x_2 - x_3 & \leq 2 \\
   -2x_1 + x_2 - 5x_3 & \geq 6
   \end{align*} \)

The above information is summarized in the following table.

**Simplex Table III**

<table>
<thead>
<tr>
<th>( c_j \rightarrow )</th>
<th>Basis</th>
<th>( c_j \rightarrow )</th>
<th>5</th>
<th>3</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_n )</td>
<td>( x_2 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \frac{5}{3} )</td>
<td>( -\frac{1}{3} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( c_n )</td>
<td>( x_1 )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>( -\frac{2}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( c_n )</td>
<td>( x_2 )</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>( -\frac{34}{5} )</td>
<td>5</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( Z = 10 )</td>
<td>( Z_j )</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>( \frac{5}{3} )</td>
<td>( 2 )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( C_j = c_j - Z_j )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( -\frac{5}{3} )</td>
<td>( -\frac{2}{3} )</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since all entries in \( C_j \)-row are either negative or zero, the optimal solution has been obtained with \( x_1 = 2, x_2 = 0 \) and Max. \( Z = 10 \).
4. Minimize \( Z = x_1 - 3x_2 + 2x_3 \)
subject to
\[
\begin{align*}
4x_1 + x_2 + x_3 & \leq 6 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

5. (i) Maximize \( Z = 5x_1 + 6x_2 + x_3 \)
subject to
\[
\begin{align*}
9x_1 + 3x_2 + 2x_3 & \leq 5 \\
4x_1 + 2x_2 + x_3 & \leq 2 \\
x_1 + 3x_2 + x_3 & \leq 3 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

(ii) Maximize \( Z = x_1 + 4x_2 - x_3 \)
subject to
\[
\begin{align*}
-5x_1 + 6x_2 - 2x_3 & \leq 30 \\
x_1 + 3x_2 + 6x_3 & \leq 12 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

6. (i) Maximize \( Z = 3x_1 + 9x_2 \)
subject to
\[
\begin{align*}
x_1 + 4x_2 & \leq 8 \\
x_1 + 2x_2 & \leq 4 \\
x_1, x_2 & \geq 0
\end{align*}
\]

(ii) Maximize \( Z = 50x_1 + 60x_2 + 80x_3 \)
subject to
\[
\begin{align*}
2x_1 + x_2 + 2x_3 & \leq 50 \\
x_1 + 6x_2 + 2x_3 & \leq 50 \\
x_1 + 2x_2 + 3x_3 & \leq 26 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

(iii) Maximize \( Z = 107x_1 + x_2 + 2x_3 \)
subject to the constraints
\[
\begin{align*}
14x_1 + x_2 - 6x_3 + 3x_4 & = 7 \\
16x_1 + 1 \cdot 2 + x_2 - 6x_4 + 5 & \leq 0 \\
3x_2 - x_1 - x_4 & \leq 0 \\
x_1, x_2, x_3, x_4 & \geq 0
\end{align*}
\]

(M.D.U. Dec. 2010)

7. A firm makes two types of furniture, chairs and tables. Profits are ₹20 per chair and ₹30 per table. Both products are processed on three machines, \( M_1, M_2, \) and \( M_3 \). The time required for each product in hours and total available in hours per week on each machine are as follows:

<table>
<thead>
<tr>
<th>Machine</th>
<th>Chair</th>
<th>Table</th>
<th>Available Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 )</td>
<td>3</td>
<td>3</td>
<td>36</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>5</td>
<td>2</td>
<td>60</td>
</tr>
<tr>
<td>( M_3 )</td>
<td>2</td>
<td>6</td>
<td>60</td>
</tr>
</tbody>
</table>

How should the manufacturer schedule his production in order to maximize profit?

(Use simplex method)

8. A company makes two kinds of leather belts. Belt A is high quality belt and belt B is of lower quality. The respective profits are ₹4 and ₹5 per belt. The production of each of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckles a day available for belt B.

### Linear Programming

What should be the daily production of each type of belt? Formulate this problem as an LP model and solve by simplex method.

9. A farmer has 1000 acres of land on which he can grow corn, wheat or soybeans. Each acre of corn costs ₹100 for preparation, requires 1 man day of work and yields a profit of ₹30. An acre of wheat costs ₹120 to prepare, requires 10 man days of work and yields a profit of ₹60. An acre of soybeans costs ₹70 to prepare, requires 8 man days of work and yields a profit of ₹20. If the farmer should be allocated to each crop to maximize profit. Formulate this as an LP model and solve it by simplex method.

10. A manufacturing firm has discontinued production of a certain unprofitable product line. This excess capacity can be used to one or more of three products 1, 2, and 3. The available capacity on machines and the number of machine-hours required for each unit of the respective product, is given below:

<table>
<thead>
<tr>
<th>Machine Type</th>
<th>Available Time (in Machine-hours per week)</th>
<th>Productivity (in Machine-hours per unit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milling Machine</td>
<td>250</td>
<td>Product 1</td>
</tr>
<tr>
<td>Lathe</td>
<td>150</td>
<td>Product 2</td>
</tr>
<tr>
<td>Grinder</td>
<td>60</td>
<td>Product 3</td>
</tr>
</tbody>
</table>

The profit per unit would be ₹20, ₹6 and ₹8 respectively for product 1, 2, and 3. Find how much of each product the firm should produce in order to maximize profit.

11. A company can make three different products A, B, and C by combining steel and rubber. Product A requires 2 units of steel and 3 units of rubber and can be sold at a profit of ₹40 per unit. Product B requires 3 units of steel and 3 units of rubber and can be sold at a profit of ₹45 per unit. Product C requires 1 unit of steel and 2 units of rubber and yields ₹24 profit per unit. There are 100 units of steel and 120 units of rubber available per day. What should be the daily production of each of the three products in order to maximize profit? Use simplex method.

### Answers

1. (i) \( x_1 = 0 \), \( x_2 = 4 \); Max. \( Z = 8 \)
(ii) \( x_1 = 2 \), \( x_2 = 4 \); Max. \( Z = 14 \)
(iii) \( x_1 = 100 \), \( x_2 = 6 \); Max. \( Z = 600 \)
(iv) \( x_1 = 24 \), \( x_2 = 14 \); Max. \( Z = 2200 \)
2. (i) \( x_1 = 5 \), \( x_2 = 0 \); Max. \( Z = 50 \)
(ii) \( x_1 = 0 \), \( x_2 = 3 \), \( x_3 = 4 \); Max. \( Z = 24 \)
(iii) \( x_1 = x_2 = 0 \), \( x_3 = 4 \); Max. \( Z = 3 \)
(iv) \( x_1 = 280 \), \( x_2 = 0 \), \( x_3 = 20 \); Max. \( Z = 2800 \)
3. \( x_1 = 0 \), \( x_2 = 4 \), \( x_3 = 2 \); Max. \( Z = 10 \)
4. \( x_1 = 4 \), \( x_2 = 5 \), \( x_3 = 0 \); Min. \( Z = -11 \)
5. (i) Unbounded solution
(ii) Unbounded solution

| Hint: When each entry in the key column is negative or zero, Z is unbounded |
6. (i) \( x_1 = 4 \), \( x_2 = 2 \); Max. \( Z = 18 \)
(ii) \( x_1 = x_2 = 0 \), \( x_3 = 23 \); Max. \( Z = 2000 \)

7. 3 chairs and 9 tables per week, maximum profit = ₹300

| Hint: Let \( x \) and \( y \) be the number of chairs and tables to be produced per week. Then Maximize \( Z = 20x + 30y \)
subject to
\[
\begin{align*}
3x + 3y & \leq 36 \\
x + 2y & \leq 50
\end{align*}
\]
8. Let $x_1$ and $x_2$ be the number of belts of type A and B respectively manufactured each day. Then the mathematical LP model is:

Maximize $Z = 4x_1 + 3x_2$

subject to

$2x_1 + x_2 \leq 1000$

$x_1 + x_2 \leq 800$

$x_1 \leq 400$

$x_2 \leq 700$

$x_1, x_2 \geq 0$

Optimal solution is $x_1 = 200, x_2 = 600$; max. $Z = \mathcal{R} 2,600$.

9. Let $x_1, x_2, x_3$ acres of land be allocated to corn, wheat and soybeans, respectively. Then the LP model is:

Maximize $Z = 30x_1 + 40x_2 + 20x_3$

subject to

$10x_1 + 12x_2 + 70x_3 \leq 1,000,000$

$7x_1 + 10x_2 + 8x_3 \leq 8000$

$x_1 + x_2 + x_3 \leq 1000$

$x_1, x_2, x_3 \geq 0$

Optimal solution is $x_1 = 250, x_2 = 625, x_3 = 0$; max. $Z = \mathcal{R} 32,500$.

10. Let $x_1, x_2, x_3$ be the number of units of products 1, 2 and 3 to be produced per week. Then the LP model is:

Maximize $Z = 20x_1 + 6x_2 + 3x_3$

subject to

$8x_1 + 2x_2 + 3x_3 \leq 250$

$4x_1 + 3x_2 \leq 150$

$2x_1 + x_3 \leq 50$

$x_1, x_2, x_3 \geq 0$

Optimal solution: $x_1 = 0, x_2 = 50, x_3 = 50$; Max. $Z = 700$.

11. 20 units of A, 20 units of B, 0 units of C with max. profit $= \mathcal{R} 1700$.

### 7.13. Artificial Variables

In the linear programming problems discussed so far, we have the following characteristics:

(i) The objective function is of maximization type

(ii) All constraints are of $\leq$ form

(iii) Right-hand side of each constraint is positive.

Adding slack variables, all inequality constraints are converted into equalities. Initial solution is found very conveniently by letting the slack variables be the initial basic variables so that each one equals the positive right-hand side of its equation.

All linear programming problems do not have the above characteristics. Now we will see what adjustments are required for other forms ($\geq$ or $< 0$) of linear programming problems.

When the constraints are of $\leq$ type but some $b_i < 0$, after adding the non-negative slack variable $s_i$, the initial solution will involve $s_i = b_i < 0$. This solution is not feasible because it violates the non-negativity condition of slack variable.

When the constraints are of $\geq$ type, after adding the non-negative surplus variable $s_i$ and letting each decision variable equal to zero, we get an initial solution $-s_i = b_i$, or $s_i = -b_i < 0$. This solution is not feasible because it violates the non-negativity condition of surplus variable.

### Linear Programming

Thus, introduction of slack/surplus variables fails to give a basic solution in many linear programming problems. Such problems are solved by the artificial variable technique.

#### 7.14. Simplex Method (Minimization Case)

Consider the general linear programming problem:

Minimize $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$

subject to the constraints

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leq b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq b_m$

and

Subtracting surplus variables $s_i$ to convert inequalities into equalities, the above problem reduces to:

Minimize $Z = c_1x_1 + c_2x_2 + ... + c_nx_n + 0x_1 + 0x_2 + ... + 0x_m$

subject to the constraints

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n - s_1 = b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n - s_2 = b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n - s_m = b_m$

and

$s_i \geq 0, \quad (i = 1, 2, ..., m)\;

An initial basic solution is obtained by assigning zero value to each decision variable $i.e.,$ by setting

$x_1 = x_2 = ... = x_n = 0$

Thus, we get

$-s_1 = b_1$ or $s_1 = -b_1$

$-s_2 = b_2$ or $s_2 = -b_2$

$\vdots$

$-s_m = b_m$ or $s_m = -b_m$

which is not feasible because it violates the non-negativity constraints $s_i \geq 0$.

\[ \therefore \text{The simplex algorithm needs modification. We now introduce } m \text{ new variables } A_1, A_2, ..., A_m \text{ into the system of constraints.} \]

These new variables are called artificial variables. The resulting system of equations can now be written as

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n - s_1 + A_1 = b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n - s_2 + A_2 = b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n - s_m + A_m = b_m$

where

$x_j \geq 0, \quad (j = 1, 2, ..., n)\;

s_j \geq 0, \quad (i = 1, 2, ..., m)$

and

$A_1 \geq 0, \quad (i = 1, 2, ..., m)$
Thus the standard LPP has been reduced to a system of \(m\) equations in \((n + 2m)\) variables \(n\) decision variables + \(m\) surplus variables + \(m\) artificial variables. An initial basic feasible solution can now be obtained by setting \((n + 2m) - m = n + m\) variables equal to zero (i.e., by setting each decision variable and each surplus variable equal to zero). Thus the initial basic feasible solution is given by \(A_1 = b_1, A_2 = b_2, \ldots, A_m = b_m\).

This, however, does not constitute a solution to the original system of equations because the two systems are not equivalent. To remove artificial variables from solution, we use

**Big-M Method or Method of Penalties**

We assign a zero co-efficient to surplus variables and a very large positive co-efficient + \(M\) to artificial variable in the objective function.

Therefore, the problem can now be re-written as follows:

**Minimize** \(Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0s_1 + 0s_2 + \ldots + 0s_m + MA_1 + MA_2 + \ldots + MA_m\)

subject to the constraints

\[
\begin{align*}
   a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - s_1 &= b_1 \\
   a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n - s_2 &= b_2 \\
   &\quad \vdots \\
   a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n - s_m &= b_m
\end{align*}
\]

and

\[
\begin{align*}
   s_j &\geq 0; \quad (j = 1, 2, \ldots, m) \\
   s_i &\geq 0; \quad (i = 1, 2, \ldots, n)
\end{align*}
\]

Solve the modified LPP by simplex method.

**Test for optimality.** Compute the values of \(C_j = c_j - Z\) in the last row of the simplex table.

(i) If all entries in \(C_j\)-row are non-negative (i.e., \(\geq 0\)), then the current basic feasible solution is optimal.

(ii) If \(C_j\) is at negative (i.e., \(C_j = \min\)) in all columns and all entries in this column are negative, then the problem is an unbounded optimal solution.

(iii) If \(C_j\) has one or more negative values, then select the variable to enter into the basis with the largest negative value, i.e., select \(x_j\) when \(C_j = \min\) (\(C_j \leq 0\)).

Determine the key row and key element in the same manner as discussed in the simplex algorithm of the maximization case.

**Remarks 1.** At any iteration of simplex method, one of the following three cases may arise:

(i) There remains no artificial variable in the basis and the optimality condition is satisfied, then the solution is an optimal basic feasible solution to the problem.

(ii) There is at least one artificial variable present in the basis with zero value and the co-efficient of \(M\) in each \(C_j\) value is non-negative, then the given LPP has no solution i.e., the current basic feasible solution is degenerate.

(iii) There is at least one artificial variable in the basis with positive value and the co-efficient of \(M\) in each \(C_j\) value is non-negative, then the given LPP has no optimal basic feasible solution. In this case, the given LPP is said to have a **pseudo optimal basic feasible solution**.

2. The artificial variables are fictitious. They are introduced only for computational purposes and do not have any physical meaning or economic significance.

3. The constraints with \(\geq\) inequality sign require a surplus variable and an artificial variable.
Some entries in $C_j$-row being negative, the current solution is not optimal.

**Step 3.** Largest negative entry in $C_j$-row is $5 - 7$ which lies in $x_1$-column. Therefore, the incoming variable is $x_1$. The ratio 2 is minimum in $A_2$-row, therefore, the outgoing basic variable is $A_2$. (In further simplex tables, we will not compute the $A_2$-column). Key element is 5.

New key row is

$$\begin{align*}
5 & x_1 & 2 & 1 & 2 & 0 & -1 & 5 & 0 \\
\end{align*}$$

**Transformation of $R_1$.** Key column entry in $R_1$ is 2.

\[
\begin{align*}
R_1(\text{new}) &= R_1(\text{old}) - 2 R_2(\text{new}) \\
12 - 2(2) &= 8 \\
2 - 2(1) &= 0 \\
4 - 2 \left( \frac{2}{5} \right) &= \frac{16}{5} \\
1 - 2(0) &= 1 \\
0 - 2 \left( \frac{1}{5} \right) &= \frac{2}{5} \\
0 - 2(0) &= 0 \\
\end{align*}
\]

**Transformation of $R_2$.** Key column entry in $R_2$ is 2.

\[
\begin{align*}
R_2(\text{new}) &= R_2(\text{old}) - 2 R_1(\text{new}) \\
10 - 2(2) &= 6 \\
2 - 2(1) &= 0 \\
2 - 2 \left( \frac{2}{5} \right) &= \frac{6}{5} \\
0 - 2(0) &= 0 \\
\end{align*}
\]

Some entries in $C_j$-row being negative, the current solution is not optimal.

**Step 4.** Largest negative entry in $C_j$-row is $1 - \frac{6}{5}$ which lies in $x_2$-column. Therefore, the incoming variable is $x_2$. The ratio $\frac{5}{2}$ is minimum in $s_1$-row, therefore, the outgoing basic variable is $s_1$. Key element is 16.

New key row is

$$\begin{align*}
3 & x_2 & 5 & 0 & 1 & \frac{16}{5} & \frac{1}{5} & 0 \\
\end{align*}$$

**Transformation of $R_3$.** Key column entry in $R_3$ is 6.

\[
\begin{align*}
R_3(\text{new}) &= R_3(\text{old}) - \frac{8}{5} R_2(\text{new}) \\
6 - \frac{6}{5} \left( \frac{2}{5} \right) &= 3 \\
0 - \frac{6}{5}(0) &= 0 \\
\end{align*}
\]
The new simplex table is given below.

### Simplex Table III

<table>
<thead>
<tr>
<th>( c_{h} )</th>
<th>Basis</th>
<th>( c_{j} \rightarrow )</th>
<th>( 5 )</th>
<th>( 3 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>( M )</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( x_{1} )</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>5/16</td>
<td>( 1/8 )</td>
</tr>
<tr>
<td>M</td>
<td>( A_{1} )</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( 5/2 )</td>
</tr>
<tr>
<td>5</td>
<td>( x_{1} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( 1/4 )</td>
</tr>
</tbody>
</table>

Since one entry in \( C_{j} \)-row is negative, the current solution is not optimal.
The new simplex table is given below.

**Simplex Table IV**

<table>
<thead>
<tr>
<th>$c_4$</th>
<th>Basis</th>
<th>$c_j \rightarrow$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$x_2$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$x_2$</td>
<td>12</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>$x_1$</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$Z = 23$

Since all entries in $C_j$ row are zero or positive, the current solution is optimal with $x_1 = 4, x_2 = 1, s_1 = 0, s_2 = 12$ and $Z = 23$.

**Example 2.** Use penalty (Big-M) method to solve the following LP problem.

Maximize $Z = x_1 + 3x_2 - 2x_3$

subject to

- $x_1 + 2x_2 - x_3 = 6$
- $x_1 - x_2 + x_3 \leq 2$
- $x_1, x_2, x_3 \geq 0$

Sol. Step 1. The constants on the right hand side of each constraint are negative. Therefore, multiplying both sides of each constraint by $-1$, we get:

- $x_1 + 2x_2 - x_3 = 6$
- $-x_1 + x_2 - x_3 \geq 2$

Introducing a surplus variable $s$ and two artificial variables $A_1$ and $A_2$, we get the standard form of the LP problem as:

Maximize $Z = x_1 + 3x_2 - 2x_3 + 0s - MA_1 - MA_2$

subject to

- $x_1 + 2x_2 + 2x_3 + A_1 = 6$
- $x_1 + x_2 - x_3 - s + A_2 = 2$

where

- $x_1, x_2, x_3, s, A_1, A_2 \geq 0$

Step 2. Since we have 2 equations in 6 variables, a solution is obtained by setting $6 - 2 = 4$ variables equal to zero and solving for the remaining 2 variables. By setting $x_1 = x_2 = x_3 = s = 0$, the initial basic feasible solution is $A_1 = 6, A_2 = 2$ and $Z = -8 M$.

The initial basic feasible solution is given in the simplex table below.

**Simplex Table I**

<table>
<thead>
<tr>
<th>$c_3$</th>
<th>Basis</th>
<th>$c_j \rightarrow$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$x_3/x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-M</td>
<td>$A_1$</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2/3 = 0.67</td>
</tr>
<tr>
<td>-M</td>
<td>$A_2$</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/2 = 0.5</td>
</tr>
<tr>
<td>Z = -8 M</td>
<td>$Z_j$</td>
<td>-2M</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Some entries in $C_j$ row being positive, the current solution is not optimal.

Step 3. Largest positive entry in $C_j$ row is $3 + 3M$ which lies in $x_2$-column. Therefore, the incoming variable is $x_2$. The ratio 2 is minimum in $A_2$-row, therefore, the outgoing basic variable is $A_2$. In further simplex tables, we will compute the $A_2$-column. Key element is 1.

New key row is

$3 \times x_2 \ 2 \ 1 \ 1 \ -1 \ -1 \ 0$

*Transformation of $R_1$. Key column entry in $R_1$ is 2.*

$R_1(\text{new}) = R_1(\text{old}) - 2R_1(\text{new})$

<table>
<thead>
<tr>
<th>$c_3$</th>
<th>Basis</th>
<th>$c_j \rightarrow$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$x_3/x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-M</td>
<td>$A_1$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1/2 = 0.5</td>
</tr>
<tr>
<td>3</td>
<td>$x_2$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

New simplex table is given below.

**Simplex Table II**

<table>
<thead>
<tr>
<th>$c_3$</th>
<th>Basis</th>
<th>$c_j \rightarrow$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$x_3/x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-M</td>
<td>$A_1$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1/2 = 0.5</td>
</tr>
<tr>
<td>3</td>
<td>$x_2$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Some entries in $C_j$ row being positive, the current solution is not optimal.
Step 4. Largest positive entry in $C_j$-row is 1 + 4M which lies in $x_2$-column. Therefore, the incoming variable is $x_2$. The ratio $\frac{1}{2}$ is minimum in $A_1$-row, therefore, the outgoing basic variable is $A_1$. In further simplex tables, we will not compute the $A_1$-column. Key element is 4.

New key row is

$$
egin{array}{c}
-2 \ x_1 \ 1/2 \ 1/4 \ 0 \ 1 \ 1/2
\end{array}
$$

Transformation of $R_2$. Key column entry in $R_2$ is $-1$.

$$
R_2(\text{new}) = R_2(\text{old}) + \frac{1}{2} R_1(\text{new})
$$

$$
\begin{array}{ll}
2 + 1(1/2) = 5/2 \\
1 + 1(-1/2) = 3/4 \\
1 + 1(0) = 1 \\
-1 + 1(1) = 0 \\
-1 + 1(1/2) = -1/2
\end{array}
$$

New simplex table is given below.

**Simplex Table III**

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>Basis</th>
<th>$c_j$ Solution $b=x_j$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s$</th>
<th>$x_2/s$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>$x_3$</td>
<td>1/2</td>
<td>-1/4</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td></td>
<td>1-&gt;</td>
</tr>
<tr>
<td>3</td>
<td>$x_3$</td>
<td>5/2</td>
<td>3/4</td>
<td>1</td>
<td>0</td>
<td>-1/2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z=13/2</td>
<td>$Z_j$</td>
<td>$c_j-Z_j$</td>
<td>11/4</td>
<td>3</td>
<td>-2</td>
<td>-5/2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One entry in $C_j$-row is positive, the current solution is not optimal.

Step 5. Largest positive entry in $C_j$-row is $5/2$ which lies in $s$-column. Therefore, the incoming variable is $s$. Outgoing basic variable is $x_2$. Key element is $1/2$.

New key row is

$$
\begin{array}{c}
0 \ s \ 1 \ -1/2 \ 0 \ 2 \ 1
\end{array}
$$

Transformation of $R_2$. Key column entry in $R_2$ is $-1/2$.

$$
R_2(\text{new}) = R_2(\text{old}) - \frac{1}{2} R_2(\text{new})
$$

$$
\begin{array}{ll}
5/2 - 1/2(1) = 3 \\
3/4 - 1/2(1) = 1/2 \\
1 - 1/2(0) = 1 \\
0 - 1/2(2) = 1 \\
-1/2 - 1/2(1) = 0
\end{array}
$$

New simplex table is given below.

**Simplex Table IV**

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>Basis</th>
<th>Solution $b=x_j$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s$</th>
<th>$x_2/s$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>s</td>
<td>1</td>
<td>-1/2</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$x_2$</td>
<td>3</td>
<td>1/2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z=9</td>
<td>$Z_j$</td>
<td>$C_j-Z_j$</td>
<td>3/2</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since all entries in $C_j$-row are negative or zero, the optimal solution has been arrived at with $x_1 = 0, x_2 = 3, x_3 = 0$ and Max $Z = 9$.

**EXERCISE 7.6**

Solve the following LP problems using Big-M method.

1. Minimize $Z = 8x + 12y$
   subject to
   $$
   \begin{align*}
   2x + 2y & \geq 1 \\
   x + 3y & \geq 2 \\
   x, y & \geq 0
   \end{align*}
   $$

2. Minimize $Z = 5x + 8y$
   subject to
   $$
   \begin{align*}
   x + y & = 5 \\
   x & \leq 4 \\
   y & \geq 2 \\
   x, y & \geq 0
   \end{align*}
   $$
14. Maximize \( Z = 3x_1 + 2x_2 + 3x_3 \) subject to \[
2x_1 + x_2 + x_3 \leq 2
\]
\[
3x_1 + 4x_2 + 2x_3 \geq 8
\]
\[
x_1, x_2, x_3 \geq 0.
\]

15. Maximize \( Z = x_1 + 2x_2 + 3x_3 - x_4 \) subject to \[
x_1 + 2x_2 + 3x_3 \leq 15
\]
\[
2x_1 + x_2 + 5x_3 \leq 20
\]
\[
x_1 + 2x_2 + x_4 \leq 10
\]
\[
x_1, x_2, x_3, x_4 \geq 0.
\]

Answers
1. \( x = 0, y = \frac{2}{3} \); Min. \( Z = 8 \)
2. \( x = 3, y = 2 \); Min. \( Z = 31 \)
3. \( x_1 = \frac{3}{5}, x_2 = \frac{6}{5} \); Min. \( Z = \frac{12}{5} \)
4. \( x = 10, y = 0 \); Min. \( Z = 30 \)
5. \( x = 5, y = 5 \); Max. \( Z = 7 \)
6. \( x = 6, y = 10 \); Max. \( Z = 2 \)
7. No optimal solution (since in the simplex tableau II, all entries in \( C_1 \) row are negative or zero so that the simplex procedure terminates. But an artificial variable with non-zero value exists in the basis).
8. \( x = 4, y = 8 \); Max. \( Z = 44 \)
9. \( x_1 = 25, x_2 = 0 \); Max. \( Z = 350 \)
10. \( x_1 = 3, x_2 = 0, x_3 = 1 \); Min. \( Z = 0 \)
11. \( x_1 = \frac{5}{4}, x_2 = 0, x_3 = \frac{9}{4} \); Min. \( Z = \frac{7}{4} \)
12. \( x_1 = 1, x_2 = 2, x_3 = 0 \); Max. \( Z = 4 \)
13. No optimal solution
14. \( x_1 = 0, x_2 = 2, x_3 = 0 \); Max. \( Z = 4 \)
15. \( x_1 = x_2 = x_3 = \frac{5}{2}, x_4 = 0 \); Max. \( Z = 15 \).

7.15. DUALITY CONCEPT

With every linear programming problem, there exists another linear programming problem called its dual since every LPP can be analyzed in two different ways without any additional data or information. For example, profit maximization problem can be seen as a problem of cost minimization and cost minimization problem can be viewed as a problem of maximizing the efficiency of using available resources. The original problem is called primal and the associated problem is called its dual. In general, either problem can be considered as primal and the other its dual.

7.16. FORMULATION OF A DUAL PROBLEM

Suppose the primal LP problem is in the form

Maximize \( Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \)

subject to the constraints

\[
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1
\]
\[
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2
\]
\[
\vdots
\]
\[
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m
\]

and

\[
x_1, x_2, \ldots, x_n \geq 0
\]
The corresponding dual problem is
Minimize $Z^* = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$
subject to the constraints

\[ a_{11} y_1 + a_{21} y_2 + \cdots + a_{1m} y_m \geq c_1 \]
\[ a_{21} y_1 + a_{22} y_2 + \cdots + a_{2m} y_m \geq c_2 \]
\[ \vdots \]
\[ a_{m1} y_1 + a_{m2} y_2 + \cdots + a_{mm} y_m \geq c_m \]

and

The characteristics of the primal and the dual problem can be stated as follows.
(i) The maximization problem in the primal becomes the minimization problem in the dual and vice versa.

(ii) The $\leq$ type constraints in the primal become $\geq$ type constraints in the dual and vice versa.

(iii) The coefficients $c_1, c_2, \ldots, c_m$ in the objective function of the primal become the constants $b_1, b_2, \ldots, b_m$ in the constraints of the dual and vice versa.

(iv) If the primal has $n$ variables and $m$ constraints, then the dual has $m$ variables and $n$ constraints. Thus the body matrix of dual is the transpose of body matrix of primal and vice versa.

(v) A new set of variables appear in the dual.

(vi) The variables in both the primal and the dual are non-negative.

(vii) The dual of the dual problem is the original primal problem itself.

### ILLUSTRATIVE EXAMPLES

**Examples 1. Obtain the dual of**

Maximize $Z = 5x_1 + 3x_2$
subject to

\[ x_1 + x_2 \leq 2 \]
\[ 3x_1 + 2x_2 \leq 10 \]
\[ 3x_1 + 8x_2 \leq 12 \]
\[ x_1, x_2 \geq 0 \]

(M.D.U. Dec. 2011)

**Sol.** (i) Primal is of maximization type
Dual will be of 'minimization' type.

(ii) Primal has 2 variables, $x_1, x_2$ and 3 constraints.
Dual will have 3 variables, $y_1, y_2, y_3$ and 2 constraints.

(iii) Primal has $\leq$ constraints.
Dual will have $\geq$ constraints.

(iv) Primal has $c_1 = 5, c_2 = 3$ and $b_1 = 2, b_2 = 10, b_3 = 12$
Dual will have $c_1 = 2, c_2 = 10, c_3 = 12$ and $b_1 = 5, b_2 = 3$

The body matrix in the primal is

\[
\begin{bmatrix}
1 & 1 \\
5 & 2 \\
3 & 8 \\
\end{bmatrix}
\]

The body matrix in the dual will be

\[
\begin{bmatrix}
1 & 0 \\
5 & 2 \\
3 & 8 \\
\end{bmatrix}
\]

Thus the dual is
Minimize $Z^* = 2y_1 + 10y_2 + 12y_3$
subject to

\[ y_1 + 5y_2 + 3y_3 \geq 5 \]
\[ y_1 + 2y_2 + 5y_3 \geq 3 \]

and

\[ y_1, y_2, y_3 \geq 0 \]

### 7.17. DUALITY PRINCIPLE

If either the primal or the dual problem has an optimal solution then so does the other problem and the optimal value of the primal's objective function is the same as that of its dual.

**Example 2. Solve the dual problem of the following LPP and hence find Max. Z:**

Maximize $Z = 20x_1 + 30x_2$
subject to

\[ 3x_1 + 3x_2 \leq 36 \]
\[ 5x_1 + 2x_2 \leq 60 \]
\[ 2x_1 + 6x_2 \leq 80 \]
\[ x_1, x_2 \geq 0 \]

**Sol.** The dual of the above LPP is:

Minimize $Z^* = 36y_1 + 50y_2 + 60y_3$
subject to

\[ 3y_1 + 5y_2 + 2y_3 \geq 20 \]
\[ 3y_1 + 2y_2 + 5y_3 \geq 30 \]
\[ y_1, y_2, y_3 \geq 0 \]

Solving this problem by simplex method, it can be verified that the final simplex table is:

<table>
<thead>
<tr>
<th>$c_A$</th>
<th>Basis</th>
<th>$c_A \rightarrow$ Solution</th>
<th>$b = (y_3)$</th>
<th>$36$</th>
<th>$50$</th>
<th>$60$</th>
<th>$0$</th>
<th>$0$</th>
<th>$M$</th>
<th>$M$</th>
</tr>
</thead>
</table>
| $y_1$ | 5     | $y_1$                     | $\begin{array}{l}
1 \\
6 \\
\end{array}$ | $0$ | $-1$ | $1$ | $2$ | $-6$ | $1$ | $-6$
| $y_3$ | 5     | $y_3$                     | $\begin{array}{l}
0 \\
-3 \\
4 \\
\end{array}$ | $1$ | $1$ | $-1$ | $-1$ | $1$ | $4$

$Z^* = 330$

$C_1 = c_1 - Z^*$

$C_1 = (5) - 330 = -325$

The optimal solution to the dual problem is

\[ y_1 = 5, y_2 = 0, y_3 = \frac{5}{2} \]

and Min. $Z^* = 330$

By duality principle = Max. $Z = \text{Min. } Z^* = 330$. 
7.18. DUAL SIMPLEX METHOD

Dual simplex method is similar to the regular simplex method which has already been discussed. The only difference lies in the criterion used for selecting the incoming and outgoing variables. In the dual simplex method, we first determine the outgoing variable and then the incoming variable whereas in the regular simplex method, the reverse is done. The regular simplex method starts with a basic feasible but non-optimal solution and works towards optimality, the dual simplex method starts with a basic infeasible but optimal solution and works towards feasibility.

7.19. WORKING PROCEDURE FOR DUAL SIMPLEX METHOD

Step 1: Check whether the objective function is to be maximized or minimized. If the objective function is to be minimized, then convert it into maximization form.

Step 2: Convert ≥ type constraints, if any, into ≤ type by multiplying such constraints by −1.

Step 3: Express all constraints as equations by adding slack variables, one for each constraint.

Step 4: Find the initial basic solution and express this information in the form of a table as in regular simplex method.

Step 5: Compute $C_j = c_j - z_j$.
(a) If all $C_j \leq 0$ and all $b_j \geq 0$, then the solution found above is the optimal basic feasible solution.
(b) If all $C_j \leq 0$ and at least one $b_j < 0$, then go to the next step.
(c) If any $C_j > 0$, the method fails.

Step 6: Selection of key row and the outgoing variable
Select the row containing the most negative $b_j$. This row is the key row and the basic variable heading the key row is the outgoing variable.

Step 7: Selection of key column and the incoming variable
(a) If all elements in the key row are $\geq 0$, then the problem does not have feasible solution.
(b) If at least one element in the key row is negative, find the ratios of the corresponding elements of $C_j$-row to these elements. Ignore the ratios associated with positive or zero elements of the key row. Choose the smallest of these ratios. The corresponding column is the key column and the variable heading the key column is the incoming variable.

Step 8: Mark the key element and make it unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.

ILLUSTRATIVE EXAMPLES

Example 1. Using dual simplex method
Maximize $Z = -3x_1 - x_2$
subject to $x_1 + x_2 \geq 1$, $2x_1 + 3x_2 \geq 2$, $x_1, x_2 \geq 0$.

**Solution:**

Step 1. The problem is that of maximization.
Step 2. Converting ≥ type constraints into ≤ type, the L.P.P. takes the form
Maximize $Z = -3x_1 - x_2$
such that $-x_1 - x_2 \leq -1$, $-2x_1 - 3x_2 \leq -2$,
subject to $x_1, x_2 \geq 0$.

Step 3. Introducing slack variables $s_1, s_2$ (one for each constraint), the problem in standard form is
Maximize $Z = -3x_1 - x_2 + s_1 + s_2$
such that $-x_1 - x_2 + s_1 = -1$,
$-2x_1 - 3x_2 + s_2 = -2$,
$x_1, x_2, s_1, s_2 \geq 0$.

Step 4. The initial basic solution is given by
$x_1 = x_2 = 0$, $s_1 = -1$, $s_2 = -2$ at which $Z = 0$.

**Dual Simplex Table I**

<table>
<thead>
<tr>
<th>$C_j$</th>
<th>Basis</th>
<th>$C_j \rightarrow$</th>
<th>-3</th>
<th>-1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$S_1$</td>
<td>$x_1$</td>
<td></td>
<td></td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$S_2$</td>
<td>$x_2$</td>
<td></td>
<td></td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Step 5. Since all $C_j \leq 0$ and all $b_j < 0$, the initial solution is optimal but infeasible. So we proceed further.

Step 6. The most negative $b_j$ (i.e., the $b_j$ which is negative and numerically largest) is $b_2 = -2$, the second row is the key row and $S_2$ is the outgoing variable.

Step 7. The key row has negative entries. The ratios of the elements in $C_j$-row to the corresponding negative elements of the key row (neglecting ratios corresponding to positive or zero elements of key row) are

\[
\begin{align*}
-3 & : 3 \\
-2 & : -3
\end{align*}
\]

Smallest ratio is $\frac{1}{3}$. The $x_2$-column is the key column and $x_2$ is the incoming variable.

Step 8. The key element is $-3$, shown circled. Drop the outgoing variable $s_2$ from the basis and introduce the incoming variable $x_2$. The new basis contains $s_1, x_2$ as the basic variables. The co-efficient of $x_2$ in the objective function is $-1$. Therefore, the entry in $C_0$ column corresponding to the new basic variable $x_2$ will be $-1$. Since the key element enclosed in the circle is not $1$, divide all elements of the key row by the key element $-3$ to make it unity and make all other elements in the key column zero. Thus the key row
is replaced by the new key row

\[
\begin{array}{ccc}
0 & -2 & -2 & -3 & 0 & 1 \\
\end{array}
\]

Now we make all other elements of the key column (i.e., \(x_2\)-column) zero.

Transformation of \(R_1\). Key column entry in \(R_1\) is \(-1\)

\[
\begin{align*}
R_1 & \text{ (new)} = R_1 \text{ (old)} - (1) R_2 \text{ (new)} \\
-1 + 1 \left( \frac{2}{3} \right) & = -\frac{1}{3} \\
-1 + 1 \left( \frac{2}{3} \right) & = -\frac{1}{3} \\
-1 + 1 (1) & = 0 \\
1 + 1 (0) & = 1 \\
0 + 1 \left( -\frac{1}{3} \right) & = -\frac{1}{3}
\end{align*}
\]

The above information is summarized in the following table:

DUAL SIMPLEX TABLE II

<table>
<thead>
<tr>
<th>(C_B)</th>
<th>Basis</th>
<th>(c_i) Solution (b = x_B)</th>
<th>(-3)</th>
<th>(-1)</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(s_1)</td>
<td>(\frac{-1}{3})</td>
<td>-(\frac{1}{3})</td>
<td>0</td>
<td>1</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>(-1)</td>
<td>(x_2)</td>
<td>(\frac{2}{3})</td>
<td>(\frac{2}{3})</td>
<td>1</td>
<td>0</td>
<td>(-\frac{1}{3})</td>
</tr>
<tr>
<td>(Z = -\frac{2}{3})</td>
<td>(Z)</td>
<td>(\frac{2}{3})</td>
<td>-1</td>
<td>0</td>
<td>(\frac{1}{3})</td>
<td></td>
</tr>
<tr>
<td>(C_j = c_j - Z_j)</td>
<td>(-\frac{7}{3})</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{3})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Step 9. Test for optimality. Here all \(C_j \leq 0\) but \(b_j = -\frac{1}{3} < 0\), the current solution is not optimal. Therefore, an improvement in the value of \(Z\) is possible and we repeat steps 6 to 8.

The first row is the key row and \(s_1\) is the outgoing variable. The ratios of the elements in \(C_j\)-row to the corresponding negative elements of the key row are

\[
\begin{align*}
-\frac{7}{3} & = 7, \frac{2}{3} = 1 \\
-\frac{3}{3} & = 1, \frac{1}{3} = 1
\end{align*}
\]

Linear Programming

The smallest ratio is 1, therefore, \(s_2\)-column is the key column and \(s_2\) is the incoming variable. The key element is \(-\frac{1}{3}\) shown circled. Drop \(s_1\) and introduce \(s_2\). The new basis contains \(s_2, x_2\) as the basic variables. Dividing all elements of the key row by the key element \(\frac{1}{3}\), the new key row is

\[
\begin{array}{cccccc}
0 & s_2 & 1 & 1 & 0 & 3
\end{array}
\]

Transformation of \(R_2\) Key column entry in \(R_2\) is \(-\frac{1}{3}\)

\[
\begin{align*}
R_2 & \text{ (new)} = R_2 \text{ (old)} - (\frac{1}{3}) R_1 \text{ (new)} \\
\frac{2}{3} + \frac{1}{3} (1) & = 1 \\
\frac{2}{3} + \frac{1}{3} (1) & = 1 \\
1 + \frac{1}{3} (0) & = 1 \\
0 + 1 \left( -\frac{1}{3} \right) & = -\frac{1}{3}
\end{align*}
\]

The above information is summarized in the following table:

DUAL SIMPLEX TABLE III

<table>
<thead>
<tr>
<th>(C_B)</th>
<th>Basis</th>
<th>(c_i) Solution (b = x_B)</th>
<th>(-3)</th>
<th>(-1)</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(s_2)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>1</td>
</tr>
<tr>
<td>(-1)</td>
<td>(x_2)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(Z = -1)</td>
<td>(Z)</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(C_j = c_j - Z_j)</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here all \(C_j \leq 0\) and all \(b_i > 0\), the optimal feasible solution has been attained. Thus the optimal feasible solution is

\[x_1 = 0, x_2 = 1, \text{ Max. } Z = -1\]

Example 2. Using dual simplex method.

Minimize \(Z = x_1 + 2x_2 + 3x_3\)

subject to \(2x_1 - x_2 + x_3 \geq 4\)
\(x_1 + x_2 + 2x_3 \leq 8\)
\(x_1 - x_2 \geq 2\)
\(x_2, x_3 \geq 0\).
Solve the problem of minimizing, converting it into a maximization problem by using the relationship:

\[
\text{Min. } Z = - \max. \ Z^* \quad \text{where } Z^* = -Z
\]

Maximize

\[
Z^* = x_1 - 2x_2 - 3x_3
\]

subject to

\[
\begin{align*}
x_1 - x_2 + x_3 &\geq 4 \\
x_1 + x_2 + 2x_3 &\leq 8 \\
x_2 - x_3 &\geq 2 \\
x_1, x_2, x_3 &\geq 0
\end{align*}
\]

Step 2. Converting ≥ type constraints into ≤ type, the L.P.P. takes the form:

Maximize

\[
Z^* = -x_1 - 2x_2 - 3x_3
\]

subject to

\[
\begin{align*}
x_1 + x_2 - x_3 &\leq 4 \\
x_1 + x_2 + 2x_3 &\leq 8 \\
x_2 + x_3 &\leq 2 \\
x_1, x_2, x_3 &\geq 0
\end{align*}
\]

Step 3. Introducing slack variables \( s_1, s_2, s_3 \), the problem in standard form is:

Maximize

\[
Z^* = x_1 - 2x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3
\]

subject to

\[
\begin{align*}
x_1 + x_2 - x_3 + s_1 &= 4 \\
x_1 + x_2 + 2x_3 + s_2 &= 8 \\
x_2 + x_3 + s_3 &= 2 \\
x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

Step 4. The initial basic solution is given by

\[
x_1 = x_2 = x_3 = 0, s_1 = -4, s_2 = 8, s_3 = -2
\]

at which \( Z^* = 0 \)

### Dual Simplex Table I

| \( C_g \) | Basis | \( C_g \rightarrow \) Solution \((b = x_g)\) | \(-1\) | \(-2\) | \(-3\) | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| 0 | \( s_1 \) | -4 | | | | 1 | 1 | 0 | 0 |
| 0 | \( s_2 \) | 8 | | | | -1 | 2 | 0 | 0 |
| 0 | \( s_3 \) | -2 | | | | 0 | -1 | 0 | 1 |

\( Z^* = 0 \)

\( C_g = c_g - Z^* \)

### Step 5. Since all \( c_g \leq 0 \) and at least one \( b_i < 0 \), the initial solution is optimal but infeasible. So we proceed further.

### Step 6. The most negative \( b_i \) is \( b_1 = -4 \), the first row is the key row and \( s_1 \) is the outgoing variable.

### Step 7. The key row has non-negative entries. The ratios of the elements in \( C_j \) row to the corresponding negative elements of the key row are

\[
\begin{align*}
\frac{-1}{-4} &= \frac{1}{1} = 1 \\
\frac{-2}{-4} &= \frac{1}{2} = \frac{1}{2}
\end{align*}
\]

### Step 8. The key element is \( -4 \), shown circled. The new basis contains \( x_1, s_2, s_3 \) as the basic variables. Since the key element is not 1, divide all elements of the key row by the key.

\[
\begin{align*}
-1 &\quad x_1 &\quad 2 &\quad 1 &\quad -1 &\quad 1 &\quad 2 &\quad 0 &\quad 0 \\
0 &\quad s_2 &\quad 6 &\quad 3 &\quad 3 &\quad 0 &\quad 2 &\quad 0 &\quad 0 \\
0 &\quad s_3 &\quad 2 &\quad 1 &\quad 2 &\quad 0 &\quad 1 &\quad 0 &\quad 1
\end{align*}
\]

Now we make all other elements of the key row zero.

#### Transformation of \( R_g \)

Key column entry in \( R_g \) is 1.

\[ R_g \rightarrow \text{new} = \frac{R_g \rightarrow \text{old}}{R_g \rightarrow \text{new}} \]

\[
\begin{align*}
8 - 1(2) &= 6, \quad 1 - 1(1) = 0, \quad 1 - 1(1) = 3 \\
2 - 1(1) &= \frac{4}{2}, \quad 0 - 1(1) = 0, \quad 1 - (1) = 1
\end{align*}
\]

The above information is summarized in the following table:

### Dual Simplex Table II

<table>
<thead>
<tr>
<th>( C_g )</th>
<th>Basis</th>
<th>( C_g \rightarrow ) Solution ((b = x_g))</th>
<th>(-1)</th>
<th>(-2)</th>
<th>(-3)</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( s_1 )</td>
<td>-4</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( s_2 )</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( s_3 )</td>
<td>-2</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

\( Z^* = -2 \)

\[
\begin{align*}
Z^* &\quad \frac{4}{2} \\
C_g &\quad c_g - Z^*
\end{align*}
\]

### Step 9. Here all \( C_g \leq 0 \) but \( b_3 = -2 < 0 \), the current solution is not optimal. So we repeat steps 6 to 8.

The third row is the key row and \( s_3 \) is the outgoing variable. The ratios of the elements in \( C_g \) row to the corresponding negative elements of the key row are

\[
\begin{align*}
-1 &\quad x_1 &\quad 2 &\quad 1 &\quad 1 &\quad \frac{1}{2} &\quad \frac{1}{2} &\quad 0 &\quad 0 \\
0 &\quad s_2 &\quad 6 &\quad 3 &\quad 3 &\quad 0 &\quad 2 &\quad 0 &\quad 0 \\
0 &\quad s_3 &\quad 2 &\quad 1 &\quad 2 &\quad 0 &\quad 1 &\quad 0 &\quad 1
\end{align*}
\]

\[
\begin{align*}
\frac{-1}{-4} &= \frac{1}{1} = 1 \\
\frac{-2}{-4} &= \frac{1}{2} = \frac{1}{2}
\end{align*}
\]

The current solution is optimal.
The $x_j$-column is the key column and $x_i$ is the incoming variable. The key element is 1, shown circled. Drop $s_2$ and introduce $x_j$. The new basis contains $x_1$, $x_2$, and $x_3$ as the basic variables. Dividing all elements of the key row by the key element 1, the new key row is:

$$
\begin{bmatrix}
-2 & 1 & 2 & 0 & 1 & -1 & 0 & 0 & -1 \\
-2 & 1 & 2 & 0 & 1 & -1 & 0 & 0 & -1 \\
\end{bmatrix}
$$

Now we make all other elements of the key column (i.e., $x_j$-column) zero.

**Transformation of $R_1$, Key column entry in $R_1$ is $-\frac{1}{2}$**

$$
R_1 \text{ (new)} = R_1 \text{ (old)} - \left(-\frac{1}{2}\right) R_2 \text{ (new)}
$$

$$
2 + \frac{1}{2} (2) = 3, \quad 1 + \frac{1}{2} (0) = 1, \quad -\frac{1}{2} + \frac{1}{2} (1) = 0 \\
\frac{1}{2} + \frac{1}{2} (-1) = 0, \quad -\frac{1}{2} + \frac{1}{2} (0) = -\frac{1}{2}, \quad 0 + \frac{1}{2} (0) = 0 \\
0 + \frac{1}{2} (-1) = -\frac{1}{2}
$$

**Transformation of $R_2$, Key column entry in $R_2$ is $\frac{3}{2}$**

$$
R_2 \text{ (new)} = R_2 \text{ (old)} - \frac{3}{2} R_3 \text{ (new)}
$$

$$
6 - \frac{3}{2} (2) = 3, \quad 0 - \frac{3}{2} (0) = 0, \quad \frac{3}{2} - \frac{3}{2} (1) = 0 \\
\frac{3}{2} - \frac{3}{2} (-1) = \frac{3}{2}, \quad \frac{3}{2} - \frac{3}{2} (0) = -\frac{3}{2}, \quad 1 - \frac{3}{2} (0) = 1 \\
0 - \frac{3}{2} (-1) = \frac{3}{2}
$$

The above information is summarized in the following table:

<table>
<thead>
<tr>
<th>Dual Simplex Table III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_B$</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>$-1$</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>$-2$</td>
</tr>
<tr>
<td>$Z^* = -7$</td>
</tr>
<tr>
<td>$C_j = c_j - Z_j^*$</td>
</tr>
</tbody>
</table>

**Step 5.** The only negative entry in $C_j$-row is $\frac{7}{8}$, which lies in $x_j$-column. Therefore the incoming variable is $x_j$. The ratio 12 is minimum in $A_{ij}$-row, therefore, the outgoing basic variable is $A_i$. In further simplex tables, we will not compute the $A_i$-column. Key element is $\frac{1}{4}$.

New key row is:

$$
\begin{bmatrix}
0 & 3 & 0 & 0 & 1 & 2 & 0 & 0 & \frac{3}{2} & 1 \\
\end{bmatrix}
$$

**Transformation of $R_1$, Key column entry in $R_1$ is $\frac{1}{8}$**

$$
R_1 \text{ (new)} = R_1 \text{ (old)} - \frac{1}{8} R_2 \text{ (new)}
$$

$$
\frac{5}{2} + \frac{1}{8} (12) = 1 \\
0 - \frac{1}{8} (0) = 0 \\
1 - \frac{1}{8} (1) = 1 \\
\frac{5}{16} + \frac{1}{8} (2) = \frac{3}{2} \\
\frac{1}{8} - \frac{1}{8} (1) = 0
$$

**Transformation of $R_2$, Key column entry in $R_2$ is $\frac{1}{4}$**

$$
R_2 \text{ (new)} = R_2 \text{ (old)} + \frac{1}{4} R_3 \text{ (new)}
$$

$$
1 + \frac{1}{4} (12) = 4 \\
1 + \frac{1}{4} (0) = 1 \\
0 + \frac{1}{4} (0) = 0 \\
\frac{1}{8} + \frac{1}{4} (2) = \frac{3}{2} \\
\frac{1}{4} + \frac{1}{4} (1) = 0
$$
The new simplex table is given below.

### Simplex Table IV

<table>
<thead>
<tr>
<th>$c_s$</th>
<th>Basis</th>
<th>$c_s \to$ Solution $b^i = x^i_p$</th>
<th>5</th>
<th>3</th>
<th>0</th>
<th>0</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$x_2$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$x_2$</td>
<td>12</td>
<td>0</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$x_1$</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Z_s$</td>
<td>5</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c_j = c_j - Z_j$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Since all entries in $C_r$-row are zero or positive, the current solution is optimal with $x_1 = 4$, $x_2 = 1$, $s_1 = 0$, $s_2 = 12$ and $\text{Min. } Z = 23$.

**Example 2.** Use penalty (Big-$M$) method to solve the following LP problem.

Maximize $Z = x_1 + 3x_2 - 2x_3$

subject to

- $-x_1 - 2x_2 - 2x_3 = -6$
- $-x_1 - x_2 + x_3 \leq -2$
- $x_p \geq 0$

**Sol. Step 1.** The constants on the right hand side of each constraint are negative. Therefore, multiplying both sides of each constraint by $-1$, we get

- $x_1 + 2x_2 + 2x_3 = 6$
- $x_1 + x_2 - x_3 \geq 2$

Introducing a surplus variable $s$ and two artificial variables $A_1$ and $A_2$, we get the standard form of the LP problem as:

Maximize $Z = x_1 + 3x_2 - 2x_3 + 0s - MA_1 - MA_2$

subject to

- $x_1 + 2x_2 + 2x_3 + A_1 = 6$
- $x_1 + x_2 - x_3 - s + A_2 = 2$

where $x_1, x_2, x_3, s, A_1, A_2 \geq 0$

and $M$ is a large positive number.

**Step 2.** Since we have 2 equations in 6 variables, a solution is obtained by setting $6 - 2 = 4$ variables equal to zero and solving for the remaining 2 variables. By setting $x_1 = x_2 = x_3 = s = 0$, the initial basic feasible solution is $A_1 = 6$, $A_2 = 2$ and $Z = -8 M$.

The initial basic feasible solution is given in the simplex table below.

### Simplex Table I

<table>
<thead>
<tr>
<th>$c_s$</th>
<th>$c_j \to$ Solution $b^i = x^i_p$</th>
<th>1</th>
<th>3</th>
<th>-2</th>
<th>0</th>
<th>-M</th>
<th>-M</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-M</td>
<td>$A_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-M</td>
<td>$A_2$</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$Z_s$</td>
<td>5</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$c_j = c_j - Z_j$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Some entries in $C_r$-row being positive, the current solution is not optimal.

**Step 3.** Largest positive entry in $C_r$-row is $3 + 3M$ which lies in $x_2$-column. Therefore, the incoming variable is $x_2$. The ratio 2 is minimum in $A_2$-row, therefore, the outgoing basic variable is $A_2$. In further simplex tables, we will not compute the $A_2$-column. Key element is 1.

New key row is

| $x_2$ | 3 | 2 | 1 | -1 | 1 | -1 | 0 |

**Transformation of $R_1$:** Key column entry in $R_1$ is 2.

<table>
<thead>
<tr>
<th>$R_1 \text{(new)} = R_1 \text{(old)} - 2R_2 \text{(new)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 - 2(2) = 2</td>
</tr>
<tr>
<td>1 - 2(-1) = 3</td>
</tr>
<tr>
<td>2 - 2(1) = 0</td>
</tr>
<tr>
<td>2 - 2(-1) = 4</td>
</tr>
<tr>
<td>0 - 2(-1) = 2</td>
</tr>
<tr>
<td>1 - 2(0) = 1</td>
</tr>
</tbody>
</table>

New simplex table is given below.

### Simplex Table II

<table>
<thead>
<tr>
<th>$c_s$</th>
<th>$c_j \to$ Solution $b^i = x^i_p$</th>
<th>1</th>
<th>3</th>
<th>-2</th>
<th>0</th>
<th>-M</th>
<th>-M</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-M</td>
<td>$A_1$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>$\frac{2 + 1}{2}$</td>
</tr>
<tr>
<td>3</td>
<td>$x_2$</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$Z_s$</td>
<td>3</td>
<td>3</td>
<td>-3</td>
<td>3</td>
<td>3</td>
<td>-3</td>
<td>-M</td>
</tr>
<tr>
<td>2</td>
<td>$c_j = c_j - Z_j$</td>
<td>-2</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Some entries in $C_j$-row being positive, the current solution is not optimal.
**Step 4.** Largest positive entry in $C_j$ row is $1 + 4M$ which lies in $x_3$ column. Therefore, the incoming variable is $x_3$. The ratio $\frac{1}{2}$ is minimum in $A_1$ - row, therefore, the outgoing basic variable is $A_1$. In further simplex tables, we will not compute the $A_1$-column. Key element is 4.

New key row is

\[-2 \quad x_3 \quad 1 \quad \frac{1}{2} \quad -\frac{1}{4} \quad 0 \quad 1 \quad \frac{1}{2}\]

**Transformation of $R_2$.** Key column entry in $R_2$ is $-1$.

\[R_2(\text{new}) = R_2(\text{old}) + \frac{1}{2} R_1(\text{new})\]

\[
\begin{align*}
2 + 1(\frac{1}{2}) &= \frac{5}{2} \\
1 + 1(-\frac{1}{4}) &= \frac{3}{4} \\
1 + 1(1) &= 1 \\
-1 + 1(0) &= 0 \\
-1 + 1(\frac{1}{2}) &= -\frac{1}{2}
\end{align*}
\]

New simplex table is given below:

**Simplex Table III**

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>Basis</th>
<th>$c_j \rightarrow$ Solution $b=x_B$</th>
<th>$I$</th>
<th>$3$</th>
<th>$-2$</th>
<th>$0$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>$x_1$</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{4}$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>1 →</td>
</tr>
<tr>
<td>$3$</td>
<td>$x_2$</td>
<td>$\frac{5}{2}$</td>
<td>$\frac{3}{4}$</td>
<td>1</td>
<td>0</td>
<td>$-\frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>$Z = \frac{13}{2}$</td>
<td>$Z_1$</td>
<td>$\frac{11}{4}$</td>
<td>$\frac{7}{4}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{5}{2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$C_j = c_j - Z_j$</td>
<td>$\frac{7}{4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One entry in $C_j$-row is positive, the current solution is not optimal.

**Step 5.** Largest positive entry in $C_j$-row is $\frac{5}{2}$ which lies in $s$-column. Therefore, the incoming variable is $s$. Outgoing basic variable is $x_3$. Key element is $\frac{1}{2}$.

New key row is

\[0 \quad s \quad 1 \quad -\frac{1}{2} \quad 0 \quad 2 \quad 1\]

**Transformation of $R_2$.** Key column entry in $R_2$ is $-1$.

\[R_2(\text{new}) = R_2(\text{old}) + \frac{1}{2} R_1(\text{new})\]

\[
\begin{align*}
5/2 + 1(1) &= 3 \\
3/4 + 1(1) &= \frac{1}{2} \\
1 + 1(0) &= 1 \\
0 + 1(2) &= 1 \\
-1/2 + 1(1) &= 0
\end{align*}
\]

New simplex table is given below:

**Simplex Table IV**

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>Basis</th>
<th>$c_j \rightarrow$ Solution $b=x_B$</th>
<th>$I$</th>
<th>$3$</th>
<th>$-2$</th>
<th>$0$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$s$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$x_2$</td>
<td>3</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$Z = 9$</td>
<td>$Z_1$</td>
<td>$Z_1$</td>
<td>$\frac{3}{2}$</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$C_j = c_j - Z_j$</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since all entries in $C_j$-row are negative or zero, the optimal solution has been arrived at with $x_1 = 0, x_2 = 3, x_3 = 0$ and Max. $Z = 9$.

**EXERCISE 7.6**

Solve the following LP problems using Big-M method.

1. Minimize $Z = 8x + 12y$
   subject to
   \[2x + 2y \leq 1 \]
   \[x + 3y \geq 2 \]
   \[x, y \geq 0 \]

2. Minimize $Z = 5x + 8y$
   subject to
   \[x + y = 5 \]
   \[x \leq 4 \]
   \[y \geq 2 \]
   \[x, y \geq 0 \]
14. Maximize \( Z = 3x_1 + 2x_2 + 3x_3 \)
subject to
\[
2x_1 + x_2 + x_3 \leq 2 \\
x_1 + 4x_2 + 2x_3 \geq 8 \\
x_1, x_2, x_3 \geq 0.
\]

15. Maximize \( Z = x_1 + 2x_2 + 3x_3 - x_4 \)
subject to
\[
x_1 + 2x_2 + 3x_3 = 15 \\
x_2 + x_3 + 5x_4 = 20 \\
x_1 + 2x_2 + x_3 + x_4 = 10 \\
x_1, x_2, x_3, x_4 \geq 0.
\]

Answers

1. \( x = 0, y = \frac{2}{3} \); Min. \( Z = 8 \)

2. \( x = 3, y = 2 \); Min. \( Z = 31 \)

3. \( x_1 = \frac{3}{5}, x_2 = \frac{6}{5} \); Min. \( Z = 12 \)

4. \( x = 10, y = 0 \); Min. \( Z = 30 \)

5. \( x = 1, y = 5 \); Max. \( Z = 7 \)

6. \( x = 6, y = 10 \); Max. \( Z = 2 \)

7. No optimal solution (since in the simplex table II, all entries in \( C_{row} \) are negative or zero so that the simplex procedure terminates. But an artificial variable with non-zero value exists in the basis).

8. \( x = 4, y = 8 \); Max. \( Z = 44 \)

9. \( x_1 = 25, x_2 = 0, x_3 = 75 \); Min. \( Z = 350 \)

10. \( x_1 = 3, x_2 = 0, x_3 = 1 \); Min. \( Z = 0 \)

11. \( x_1 = \frac{5}{4}, x_2 = 0, x_3 = \frac{9}{4} \); Min. \( Z = 7 \)

12. \( x_1 = 5, x_2 = 0, x_3 = 5 \); Max. \( Z = 4 \)

13. No optimal solution

14. \( x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0 \); Max. \( Z = 15 \)

7.15. DUALITY CONCEPT

With every linear programming problem, there exists another linear programming problem called its dual since every LPP can be analyzed in two different ways without any additional data or information. For example, profit maximization problem can be seen as a problem of cost minimization and cost minimization problem can be viewed as a problem of maximizing efficiency using available resources. The original problem is called primal and the associated problem is called its dual. In general, either problem can be considered as primal and the other its dual.

7.16. FORMULATION OF A DUAL PROBLEM

Suppose the primal LP problem is in the form

Maximize \( Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \)

subject to the constraints

\[
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\
x_1, x_2, \ldots, x_n \geq 0
\]

and

\[
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_0 \\
x_1, x_2, \ldots, x_n \geq 0
\]
The corresponding dual problem is

Minimize \( Z^* = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m \)

subject to the constraints

\[
\begin{align*}
& a_{11} y_1 + a_{12} y_2 + \ldots + a_{1m} y_m \geq c_1 \\
& a_{21} y_1 + a_{22} y_2 + \ldots + a_{2m} y_m \geq c_2 \\
& \vdots \\
& a_{n1} y_1 + a_{n2} y_2 + \ldots + a_{nm} y_m \geq c_n \\
\end{align*}
\]

and

\[ y_1, y_2, \ldots, y_m \geq 0 \]

and the characteristics of the primal and the dual problem can be stated as follows:

(i) The maximization problem in the primal becomes the minimization problem in the dual and vice versa.

(ii) The \( \leq \) type constraints in the primal become \( \geq \) type constraints in the dual and vice versa.

(iii) The coefficients \( c_1, c_2, \ldots, c_n \) in the objective function of the primal become the constants \( b_1, b_2, \ldots, b_m \) in the constraints of the dual and vice versa.

(iv) If the primal has \( n \) variables and \( m \) constraints, then the dual has \( m \) variables and \( n \) constraints. Thus the body matrix of dual is the transpose of body matrix of primal and vice versa.

(v) A new set of variables appear in the dual.

(vi) The variables in both the primal and the dual are non-negative.

(vii) The dual of the dual problem is the original primal problem itself.

---

**ILLUSTRATIVE EXAMPLES**

**Examples 1.** Obtain the dual of

Maximize \( Z = 5x_1 + 3x_2 \)

subject to

\[
\begin{align*}
x_1 + x_2 & \leq 2 \\
5x_1 + 2x_2 & \leq 10 \\
3x_1 + 8x_2 & \leq 12 \\
x_1, x_2 & \geq 0 \\
\end{align*}
\]

**Sol.** (i) Primal is of maximization type

Dual will be of minimization type.

(ii) Primal has 2 variables \( x_1, x_2 \) and 3 constraints.

Dual will have 3 variables \( y_1, y_2, y_3 \) and 2 constraints.

(iii) Primal has \( \leq \) constraints.

Dual will have \( \geq \) constraints.

(iv) Primal has \( c_1 = 5, c_2 = 3 \) and \( b_1 = 2, b_2 = 10, b_3 = 12 \)

Dual will have \( c_1 = 2, c_2 = 10, c_3 = 12 \) and \( b_1 = 5, b_2 = 3 \)

The body matrix in the primal is

\[
\begin{bmatrix}
1 & 1 \\
5 & 2 \\
3 & 8
\end{bmatrix}
\]

---

**7.17. DUALITY PRINCIPLE**

If either the primal or the dual problem has an optimal solution then so does the other problem and the optimal value of the primal’s objective function is the same as that of its dual.

**Example 2.** Solve the dual problem of the following LPP and hence find \( \text{Max. } Z \):

Maximize \( Z = 20x_1 + 30x_2 \)

subject to

\[
\begin{align*}
x_1 + 5y_1 + 3y_2 & \leq 5 \\
y_1 + 2y_2 + 8y_3 & \leq 3 \\
y_1, y_2, y_3 & \geq 0 \\
\end{align*}
\]

**Sol.** The dual of the above LPP is:

Minimize \( Z^* = 36y_1 + 50y_2 + 60y_3 \)

subject to

\[
\begin{align*}
3x_1 + 3y_2 & \leq 36 \\
5x_1 + 2x_2 & \leq 50 \\
2x_1 + 6x_2 & \leq 60 \\
x_1, x_2, x_3 & \geq 0 \\
\end{align*}
\]

Solving this problem by simplex method, it can be verified that the final simplex table is:

<table>
<thead>
<tr>
<th>( c_B )</th>
<th>Basis</th>
<th>( c_j \rightarrow ) Solution ( b = (x_B) )</th>
<th>36</th>
<th>50</th>
<th>60</th>
<th>0</th>
<th>0</th>
<th>M</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>( x_1 )</td>
<td>5</td>
<td>1</td>
<td>13</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>60</td>
<td>( y_3 )</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>-3</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( Z^* = 330 )</td>
<td>( y_1 )</td>
<td>0</td>
<td>17</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>9</td>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

The optimal solution to the dual problem is

\[
y_1 = 5, y_2 = 0, y_3 = 0 \quad \text{and } \quad \text{Min. } Z^* = 330
\]

By duality principle = \( \text{Max. } Z = \text{Min. } Z^* = 330 \).
7.18. DUAL SIMPLEX METHOD

DUAL SIMPLEX METHOD

The dual simplex method is similar to the regular simplex method which has already been discussed. The only difference lies in the criterion used for selecting the incoming and outgoing variable. In the dual simplex method, the reverse is done. The regular simplex method starts with a basic feasible but non-optimal solution and works towards optimality, the dual simplex method starts with a basic infeasible but optimal solution and works towards feasibility.

7.19. WORKING PROCEDURE FOR DUAL SIMPLEX METHOD

Step 1: Check whether the objective function is to be maximized or minimized. If the objective function is to be minimized, then convert it into maximization form.

Step 2: Convert ≥ type constraints, if any, into ≤ type by multiplying such constraints by -1.

Step 3: Express all constraints as equations by adding slack variables, one for each constraint.

Step 4: Find the initial basic solution and express this information in the form of a table as in regular simplex method.

Step 5: Compute \( C_j - Z_j \)

(a) If all \( C_j \leq 0 \) and all \( b_i \geq 0 \), the solution found above is the optimal basic feasible solution.

(b) If all \( C_j \leq 0 \) and at least one \( b_i < 0 \), then go to the next step.

(c) If any \( C_j > 0 \), the method fails.

Step 6: Selection of key row and the outgoing variable

Select the row containing the most negative \( b_i \). This is the key row and the basic variable heading the key row is the outgoing variable.

Step 7: Selection of key row and the incoming variable

(a) If all elements in the key row are \( \geq 0 \), then the problem does not have feasible solution.

(b) If at least one element in the key row is negative, find the ratios of the corresponding elements of \( C_j \)-row to these elements. Ignore the ratios associated with positive or zero elements of the key row. Choose the smallest of these ratios. The corresponding column is the key column and the variable heading the key column is the incoming variable.

Step 8: Mark the key element and make it unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.

ILLUSTRATIVE EXAMPLES

Example 1. Using dual simplex method

Maximize \( Z = -3x_1 - x_2 \)
subject to \( x_1 + x_2 \geq 1, 2x_1 + 3x_2 \geq 2, x_1, x_2 \geq 0 \).

is replaced by the new key row
\[ \begin{align*}
-1 & \quad 2 \\
x_3 & \quad 2 \\
\frac{2}{3} & \quad 1 \\
0 & \quad 0 \\
\frac{1}{3} & \quad -1 
\end{align*} \]

Now we make all other elements of the key column (i.e., \( x_2 \)-column) zero.

**Transformation of \( R_1 \), Key column entry in \( R_1 \) is \(-1\)**

\[ R_1 \text{ (new)} = R_1 \text{ (old)} - (-1) \cdot R_2 \text{ (new)} \]

\[ \begin{align*}
-1 + 1 & = 0 \\
\frac{2}{3} & = \frac{1}{3} \\
\frac{1}{3} & = \frac{1}{3} \\
0 & = 0 \\
\frac{1}{3} & = \frac{1}{3}
\end{align*} \]

The above information is summarized in the following table:

**Dual Simplex Table II**

<table>
<thead>
<tr>
<th>( C_B )</th>
<th>( C_j )</th>
<th>( s_j )</th>
<th>( \frac{1}{3} )</th>
<th>( -1 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( s_1 )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>( x_2 )</td>
<td>( \frac{2}{3} )</td>
<td>( \frac{2}{3} )</td>
<td>1</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>( Z = \frac{2}{3} )</td>
<td>( Z_j )</td>
<td>( \frac{2}{3} )</td>
<td>-1</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>( C_j = c_j - Z_j )</td>
<td>( \frac{7}{3} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
<td></td>
</tr>
</tbody>
</table>

**Step 9. Test for optimality.** Here all \( C_j \leq 0 \) but \( b_1 = -\frac{1}{3} < 0 \), the current solution is not optimal. Therefore, an improvement in the value of \( Z \) is possible and we repeat steps 6 to 8.

The first row is the key row and \( s_1 \) is the outgoing variable. The ratios of the elements is \( C_j \)-row to the corresponding negative elements of the key row are

\[ \begin{align*}
-\frac{7}{3} & = 1 \\
\frac{1}{3} & = 1 \\
\frac{1}{3} & = 1
\end{align*} \]

**Example 2. Using dual simplex method.**

Minimise \( Z = x_1 + 2x_2 + 3x_3 \)

subject to

\[ \begin{align*}
2x_1 - x_2 + x_3 & \geq 4 \\
x_1 + x_2 + 2x_3 & \leq 8 \\
x_2 - x_3 & \geq 2 \\
x_1, x_2, x_3 & \geq 0.
\end{align*} \]
Sol.

**Step 1.** The problem is that of minimization. Converting it into maximization problem by using the relationship

Min.  \( Z = -\max. Z^* \), where \( Z^* = -Z \), we have

Maximize \( Z^* = -x_1 - 2x_2 - 3x_3 \)

subject to

\[
\begin{align*}
2x_1 + x_2 + x_3 & \geq 4 \\
x_1 + x_2 + 2x_3 & \leq 8 \\
x_2 - x_3 & \leq 2 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

**Step 2.** Converting \( \geq \) type constraints into \( \leq \) type, the L.P.P. takes the form

Maximize \( Z^* = -x_1 - 2x_2 - 3x_3 \)

subject to

\[
\begin{align*}
-2x_1 + x_2 - x_3 & \leq 4 \\
x_1 + x_2 + 2x_3 & \leq 8 \\
-x_2 + x_3 & \leq 2 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

**Step 3.** Introducing slack variables \( s_1, s_2, s_3 \), the problem in standard form is:

Maximize \( Z^* = -x_1 - 2x_2 - 3x_3 + s_1 + 0s_2 + 0s_3 \)

subject to

\[
\begin{align*}
-2x_1 + x_2 - x_3 + s_1 & = 4 \\
x_1 + x_2 + 2x_3 + s_2 & = 8 \\
-x_2 + x_3 + s_3 & = 2 \\
x_1, x_2, s_1, s_2, s_3 & \geq 0
\end{align*}
\]

**Step 4.** The initial basic solution is given by

\[
\begin{align*}
x_1 = x_2 = x_3 = 0, \quad x_i = -4, \quad s_2 = 8, \quad s_3 = 2
\end{align*}
\]

at which \( Z^* = 0 \)

**Dual Simplex Table I**

<table>
<thead>
<tr>
<th>( C_x )</th>
<th>Basis</th>
<th>( C \rightarrow )</th>
<th>Solution ( b = x_p )</th>
<th>-1</th>
<th>-2</th>
<th>-3</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( s_1 )</td>
<td>-4</td>
<td>( x_1 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( s_2 )</td>
<td>1</td>
<td>( x_2 )</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( s_3 )</td>
<td>0</td>
<td>( x_3 )</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( Z^* )</td>
<td>( C = c_j - Z^* )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Step 5.** Since all \( C_j \leq 0 \) and at least one \( b_i < 0 \), the initial solution is optimal but infeasible. So we proceed further.

**Step 6.** The most negative \( b_i \) is \( b_i = -4 \), the first row is the key row and \( s_1 \) is the outgoing variable.

**Step 7.** The key row has negative entries. The ratios of the elements in \( C_j \) row to the corresponding negative elements of the key row are

\[
\begin{align*}
\frac{-1}{-2} &= \frac{1}{2} = \frac{-3}{-1} = 3
\end{align*}
\]

**Step 8.** The key element is \( -2 \), shown circled. The new basis contains \( x_1, s_2, s_3 \) as the basic variables. Since the key element is not 1, divide all elements of the key row by the key element

\[
\begin{align*}
-1 & \rightarrow x_1 \\
2 & \rightarrow 1 \\
-1 & \rightarrow \frac{1}{2} \\
2 & \rightarrow 1 \\
-1 & \rightarrow \frac{1}{2} \\
0 & \rightarrow 0 \\
0 & \rightarrow 0 \\
0 & \rightarrow 0 \\
0 & \rightarrow 0
\end{align*}
\]

Now we make all other elements of the key column zero.

**Transformation of \( R_1 \):** Key column entry in \( R_1 \) is 1.

\[
R_1 (\text{new}) = R_1 (\text{old}) - 2 R_2 (\text{new})
\]

\[
\begin{align*}
8 - 1 & (2) = 6, \quad 1 - 1 (1) = 0, \quad 1 - 1 (\frac{1}{2}) = \frac{3}{2} \\
2 & - 1 (\frac{1}{2}) = \frac{3}{2}, \quad 0 - 1 (\frac{1}{2}) = \frac{1}{2}, \quad 1 - 1 (0) = 1 \\
0 & - 1 (0) = 0
\end{align*}
\]

**Transformation of \( R_2 \):** Key column entry in \( R_2 \) is 0.

\[
R_2 (\text{new}) = R_2 (\text{old})
\]

The above information is summarized in the following table:

**Dual Simplex Table II**

<table>
<thead>
<tr>
<th>( C_x )</th>
<th>Basis</th>
<th>( C \rightarrow )</th>
<th>Solution ( b = x_p )</th>
<th>-1</th>
<th>-2</th>
<th>-3</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>( x_1 )</td>
<td>2</td>
<td>( x_2 )</td>
<td>1</td>
<td>( x_3 )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( s_2 )</td>
<td>6</td>
<td>0</td>
<td>( x_1 )</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( s_3 )</td>
<td>0</td>
<td>( x_2 )</td>
<td>0</td>
<td>( x_3 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( Z^* )</td>
<td>( C = c_j - Z^* )</td>
<td>0</td>
<td>( x_1 )</td>
<td>0</td>
<td>( x_2 )</td>
<td>0</td>
<td>( x_3 )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Step 9.** Here all \( C_j \leq 0 \) but \( b_3 = -2 \), the current solution is not optimal. So we repeat steps 6 to 8.

The third row is the key row and \( s_3 \) is the outgoing variable. The ratios of the elements in \( C_j \) row to the corresponding negative elements of the key row are

\[
\begin{align*}
-2 & \rightarrow 5, \quad \frac{5}{2} = 5 \\
0 & \rightarrow -1, \quad \frac{1}{2} = 1 \quad \frac{1}{2} = 1
\end{align*}
\]
The $x_2$-column is the key column and $x_2$ is the incoming variable. The key element is -1, shown circled. Drop $x_2$ and introduce $x_2$. The new basis contains $x_1$, $x_2$, $x_3$ as the basic variables. Dividing all elements of the key row by the key element -1, the new key row is variables. Now we make all other elements of the key column (i.e., $x_2$-column) zero.

Transformation of $R_1$. Key column entry in $R_1$ is $-\frac{1}{2}$.

\[
R_1 \text{(new)} = R_1 \text{(old)} - \left(-\frac{1}{2}\right) R_3 \text{(new)}
\]

\[
2 + \frac{1}{2} (2) = 3, \quad 1 + \frac{1}{2} (0) = 1, \quad -\frac{1}{2} + \frac{1}{2} (1) = 0
\]

\[
\frac{1}{2} (1) = 0, \quad -\frac{1}{2} + \frac{1}{2} (0) = -\frac{1}{2}, \quad 0 + \frac{1}{2} (0) = 0
\]

\[0 + \frac{1}{2} (1) = \frac{1}{2}
\]

Transformation of $R_2$. Key column entry in $R_2$ is $\frac{3}{2}$.

\[
R_2 \text{(new)} = R_2 \text{(old)} - \left(-\frac{3}{2}\right) R_3 \text{(new)}
\]

\[
6 - \frac{3}{2} (2) = 3, \quad 0 - \frac{3}{2} (0) = 0, \quad \frac{3}{2} \frac{3}{2} (1) = 0
\]

\[
\frac{3}{2} (1) = 3, \quad \frac{3}{2} \frac{3}{2} (0) = \frac{3}{2}, \quad \frac{3}{2} \frac{3}{2} (0) = 1
\]

\[0 - \frac{3}{2} (1) = \frac{3}{2}
\]

The above information is summarized in the following table:

**Dual Simplex Table III**

<table>
<thead>
<tr>
<th>$C_B$</th>
<th>Basis</th>
<th>$C_j \rightarrow$ Solution</th>
<th>$b (= x_0)$</th>
<th>$-1$</th>
<th>$-2$</th>
<th>$-3$</th>
<th>$0$</th>
<th>$0$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>$x_1$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$s_2$</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>-2</td>
<td>$x_2$</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$Z^* = -7$</td>
<td>$Z^*_f$</td>
<td>-1</td>
<td>-2</td>
<td>2</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{5}{2}$</td>
<td></td>
</tr>
<tr>
<td>$C_j = c_j - Z^*_f$</td>
<td>$0$</td>
<td>0</td>
<td>-5</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$-\frac{5}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX—1

SHORT ANSWER TYPE QUESTIONS

FOURIER SERIES

1. The period of \( \sin 4x \) is .......
2. The period of \( \cos nx \) is .......
3. The period of \( |\sin t| \) is .......
4. The period of \( f(x) = \cos x + \frac{1}{2} \cos 2x + \frac{1}{3} \cos 3x \) is .......
5. The period of a constant function is .......
6. The period of \( f(x) = \cos \left( \frac{4n \pi x}{3\lambda} \right) \) is .......
7. Fourier expansion of an odd function \( f(x) \) in \((-\pi, \pi)\) has only ....... terms.
8. If \( f(x) \) is an odd function in \((-\pi, \pi)\), then write the values of \( a_0, a_n \), and \( b_n \).
9. Fourier expansion of an even function \( f(x) \) in \((-\pi, \pi)\) has only ....... terms.
10. If \( f(x) \) is an even function in \((-\pi, \pi)\), then write the values of \( a_0, a_n \), and \( b_n \).
11. In the Fourier series expansion of \( f(x) = \left( \frac{\pi - x^2}{2} \right) \) in \((0, 2\pi)\), the value of \( a_n \) is .......
12. In the Fourier series expansion of \( f(x) = x \sin x \) in \((0, 2\pi)\), the value of \( a_n \) is .......
13. Write the Fourier series for \( f(x) = x^2 \) in \((-\pi, \pi)\).
14. If \( f(x) = x^2 \) in \((-\pi, \pi)\), then the Fourier series of \( f(x) \) contains only ....... terms.
15. In the Fourier series of \( f(x) = |x| \) in \((-\pi, \pi)\), the value of \( b_n \) is .......
16. Express \( f(x) = |x|, -\pi < x < \pi \), as Fourier series.
17. In the Fourier series expansion of \( f(x) = x \sin x \) in \((-\pi, \pi)\), the value of \( b_4 \) is .......
18. In the Fourier series expansion of \( f(x) = x \) in \((-\pi, \pi)\), the ....... terms are absent.
19. Expand in a Fourier series the function \( f(x) = x \) in the interval \( 0 < x < 2\pi \).
20. Expand in a Fourier series the function \( f(x) = \frac{1}{2} x \) in the interval \((-\pi, \pi)\).
21. In the Fourier series expansion of \( f(x) = |\sin x| \) in \((-\pi, \pi)\), the value of \( b_n \) is .......
22. The expansion of \( f(x) = |\cos x| \) as a Fourier series in \((-\pi, \pi)\) does not contain ....... terms.
23. In the Fourier expansion of \( f(x) = x \cos x \) in \((-\pi, \pi)\), the ....... terms are absent.
24. The Fourier series expansion of \( f(x) = \sqrt{1 - \cos x} \) in \((-\pi, \pi)\) contains only ....... terms.
25. Is the function \( f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases} \) even or odd?
26. Is the following function even or odd?
\[ f(x) = \begin{cases} 1+2x & -\pi < x < 0 \\ \frac{1}{x} & 0 < x < \pi \end{cases} \]

27. If \( f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases} \), then \( f(x) \) is an ..... function in \((-\pi, \pi)\).

28. If \( f(t) = \begin{cases} 1 & -1 < t < 0 \\ -1 & 0 < t < 1 \end{cases} \), then \( f(t) \) is an ..... function in \((-1, 1)\).

29. Dirichlet conditions for the expansion of a function as a Fourier series in \([a, b]\) are ..... 

30. If \( f(x) = x^2 \) in \((-1, 1)\), then the Fourier co-efficient \( b_n \) = ..... 

31. If \( f(x) \) is an even function in \((-\pi, \pi)\), then the graph of \( f(x) \) is symmetrical about ..... 

32. If \( x = c \) is a point of discontinuity then the Fourier series of \( f(x) \) at \( x = c \) gives \( f(x) = ..... \)

33. If \( f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases} \), then \( f(0) = ..... \)

34. If \( f(x) \) is a periodic function with period \( 2T \), then the value of Fourier co-efficient \( a_n \) = ..... 

35. The mean value of \( f(x) \cos nx \) in \((0, 2\pi)\) = ..... 

36. If \( f(x) \) is an odd function in \((-1, 1)\), then the Fourier co-efficient \( a_n \) = ..... 

37. Find the Fourier series to represent \( f(x) \) = \( x^2 - 2 \) when \(-2 \leq x \leq 2\).

38. The half-range cosine series for a function \( f(x) \) in the interval \((0, l)\) is ..... 

39. The half-range sine series for a function \( f(x) \) in the interval \((0, \pi)\) is ..... 

40. The half-range sine series for 1 in \((0, \pi)\) is ..... 

**FOURIER TRANSFORMS**

41. Fourier sine integral of \( f(x) \) is ..... 

42. Fourier cosine integral of \( f(x) \) is ..... 

43. Fourier integral of \( f(x) \) is ..... 

44. Fourier cosine transform of \( f(x) \) in \((0, \infty)\) is ..... 

45. The inversion formula for the Fourier cosine transform of \( F(x) \) is ..... 

46. Fourier sine transform of \( f(x) \) in \((0, \infty)\) is ..... 

47. The inverse Fourier sine transform of \( F(s) \) is ..... 

48. Fourier transform of \( f(x) \) is ..... 

49. The inverse Fourier transform of \( F(s) \) is ..... 

50. The finite Fourier sine transform of \( f(x) \) in \(0 < x < c\) is defined as ..... 

51. The infinite Fourier sine transform of \( F(x) \) is given by ..... 

52. The finite Fourier cosine transform of \( f(x) \) in \(0 < x < c\) is defined as ..... 

53. The inverse finite Fourier cosine transform of \( F(x) \) is given by ..... 

54. Write the kernel of each of the following transforms:
   (i) Fourier sine transform
   (ii) Fourier cosine transform
   (iii) Fourier transform.

Appendix I

55. Find the Fourier sine transform of \( e^{-|x|} \).

56. Find the Fourier cosine transform of \( \frac{1}{x} \).

57. What is the Fourier cosine transform of \( \frac{1}{\sqrt{x}} \)?

58. Find the Fourier cosine transform of \( f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x > a \end{cases} \).

59. Find the Fourier transform of \( f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases} \).

60. Inverse finite Fourier transform of \( F(x) = \frac{1}{\pi} \frac{\sin \pi x}{\pi^2 x^2} \) where \( 0 \leq x \leq 1 \) is ..... 

61. Find the Fourier sine transform of \( f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & x \geq \pi \end{cases} \).

62. Find the Fourier cosine transform of \( f(x) = e^{ax} \), \( a > 0 \).

63. Write the finite Fourier cosine transform of \( f(x) \) for \( x \) in \((0, \pi)\).

64. If \( f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ 0 & x > \pi \end{cases} \), then Fourier cosine integral of \( f(x) \) is ..... 

65. Is Fourier transform a linear operation?

66. If \( F(x) \) and \( G(x) \) are the Fourier cosine transforms of \( f(x) \) and \( g(x) \) respectively, then \( F(x) \) for \( f(x) + g(x) = ..... \), where \( a \) and \( b \) are constants.

67. If \( F(x) \) is the Fourier transform of \( f(x) \), then the Fourier transform of \( f(ax) \) is ..... 

68. If \( F(x) = \lambda F(x) \), then \( \lambda = ..... \)

69. If \( F(x) \) is the complex Fourier transform of \( f(x) \), then \( F(x) \) for \( f(x) \) is ..... 

70. If \( F(x) \) is the complex Fourier transform of \( f(x) \), then \( F(x) \) for \( f(x) \) is ..... 

71. If \( F(x) \) and \( G(x) \) are Fourier sine and cosine transforms of \( f(x) \) respectively, then \( F(x) \) for \( f(x) \) is ..... 

72. If the Fourier transform of \( e^{ax} \) is \( \frac{1}{\lambda} e^{-\lambda x} \), write the Fourier transform of \( e^{-x^2} \) is ..... 

73. State convolution theorem for Fourier transforms in words as well as in symbols.

74. Write the relation between Fourier and Laplace transforms.

75. Write the Fourier transform of dirac-delta function.

**FUNCTIONS OF A COMPLEX VARIABLE**

76. Real part of \( e^{\frac{3}{2} + i\theta} \) is ..... 

77. Imaginary part of \( e^{6 + 2i\theta} \) is ..... 

78. Real part of \( e^{ix} \) is ..... 

79. \( e^{ix} \) is a periodic function with period .....
80. Find the real and imaginary parts of $\exp(z^2)$.

81. $\text{Re} \left[ \log (a + ib) \right] = \ldots$, $\text{Im} \left[ \log (a + ib) \right] = \ldots$

82. Find the general value of $\log (-3)$.

83. Separate $\log (4 + 3i)$ into real and imaginary parts.

84. Prove that $\tan \left( i \log \frac{a - ib}{a + ib} \right) = \frac{2ab}{a^2 - b^2}$.

85. Is $\bar{z}$ wholly real or purely imaginary?

86. Prove that $\text{Log} \bar{z} = -\left( \frac{2\pi}{3} + \frac{1}{2} \right)z$.

87. Imaginary part of $\log (1 + i \tan \alpha)$ is $\ldots$

88. Period of tanh $x$ is $\ldots$

89. Real part of sin $z$ is $\ldots$

90. Imaginary part of $\cos \bar{z}$ is $\ldots$

91. Real part of tan $(x + iy)$ is $\ldots$

92. Imaginary part of $\cot \bar{z}$ is $\ldots$

93. Real part of sec $z$ is $\ldots$

94. Imaginary part of $\sec \bar{z}$ is $\ldots$

95. Real part of sinh $(x + iy)$ is $\ldots$

96. Imaginary part of $\cosh (x + iy)$ is $\ldots$

97. $\sinh^{-1} x = \log (\ldots)$.

98. $\tanh^{-1} x = \frac{1}{2} \log (\ldots)$.

99. If $a + ib = \tan^{-1} (x + iy)$ then $a = \ldots$

100. If $c \tan (x + iy) = a + ib$, prove that $\tan 2x = \frac{2ca}{c^2 - a^2 - b^2}$.

101. Find $\tan h x$ if $5 \sinh x - \cosh x = 5$.

102. Show that $\lim_{z \to 0} \frac{\text{Log} \bar{z} - \text{Log} z}{z^2}$ does not exist.

103. Show that the function $f(z)$ defined by

$$f(z) = \begin{cases} \text{Re} \left( \frac{z}{|z|^2} \right), & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not continuous at $z = 0$.

104. Show that the function

$$f(z) = \begin{cases} \text{Im} \left( \frac{z}{|z|^2} \right), & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not continuous at $z = 0$.

105. If $f(z) = u(x, y) + iv(x, y)$ is analytic, then write $f'(z)$ in terms of

(i) partial derivatives of $u$ and $v$,

(ii) partial derivatives of $u$ only

(iii) partial derivatives of $v$ only

106. The C-R equations for $f(z) = u(x, y) + iv(x, y)$ to be analytic are $\ldots$

107. Write the polar form of Cauchy-Riemann equations.

108. What is a harmonic function?
137. Evaluate \( \frac{dz}{5z-7} \), where \( C \) is \( |z| = 1 \).

138. Evaluate \( \int \frac{e^{-z}}{z+1} \, dz \), where \( C \) : \( |z| = 2 \).

139. Evaluate \( \int \frac{e^z}{(z-2)^2} \, dz \), where \( C \) : \( |z| = 1 \).

140. Evaluate \( \int \frac{z^2 + z + 1}{z - 2z^2 - 3z + 2} \, dz \), where \( C \) is the ellipse \( 4x^2 + 9y^2 = 1 \).

141. Evaluate \( \int \frac{5z^3 - 7z^2 + 3x + 2}{z - 4} \, dz \), where \( C \) : \( 9x^2 + 4y^2 = 36 \).

142. Evaluate \( \int \frac{\cos z}{z - \pi} \, dz \), where \( C \) : \( |z - 1| = 3 \).

143. The value of \( \int_C \frac{dz}{z-\pi} \), where \( C \) is the contour represented by the straight line from \( z = -i \) to \( z = i \), is ...... \( \)

144. The value of \( \int_C \frac{z \, dz}{\sin z} \), where \( C \) : \( |z| = 4 \), is ...... \( \)

**POWER SERIES AND CONTOUR INTEGRATION**

145. Expand the function \( \frac{\sin z}{z-\pi} \) about \( z = \pi \).

146. Write Taylor’s series for \( \frac{1}{z-2} \), \( |z| > 2 \).

147. Taylor’s series expansion of \( \log (1 + z) \) about \( z = 0 \) is ...... \( \)

148. Taylor’s series expansion of \( \frac{1}{z-2} \) in \( |z| < 1 \) is ...... \( \)

149. Show that \( z^2 + 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n \) when \( |z+1| < 1 \).

150. Taylor’s series expansion of \( \cos z \) about \( z = \frac{\pi}{4} \) is ...... \( \)

151. Expand \( \frac{1}{z^2 - 3z + 2} \) in the region \( |z| < 1 \).

152. Write Laurent’s series for \( \frac{1}{z^2 - 3z + 2} \) when \( |z| > 2 \).

153. Write Laurent’s series for \( \frac{e^z}{(z-1)^2} \) about the singularity \( z = 1 \).

154. The coefficient of \( (z - a)^{-1} \) in the expansion of \( f(z) \) about an isolated singularity \( z = a \) is called ...... \( \)

**APPENDIX—I**

155. If \( f(z) \) has a simple pole at \( z = a \), then \( \text{Res} \, f(z), a = \ldots \).

156. If \( f(z) \) has a pole of order \( m \) at \( z = a \), then \( \text{Res} \, f(z), a = \ldots \).

157. If \( f(z) = \frac{a(z)}{b(z)} \) has a simple pole at \( z = a \), then \( \text{Res} \, f(z), a = \ldots \).

158. Find the poles of the function \( f(z) = \frac{z^2}{z^2 + z} \) inside the circle \( |z| = 2 \).

159. Find the poles of the function \( f(z) = \frac{e^z}{\cos z} \) inside the unit circle \( |z| = 1 \).

160. The poles of \( \cot z \) are ...... \( \)

161. If \( f(z) \) has a pole of order \( 3 \) at \( z = a \), then \( \text{Res} \, f(z), a = \ldots \).

162. The singularity of \( f(z) = \frac{z^2 + 2}{(z-3)^2} \) is ...... \( \)

163. Poles of \( f(z) = \frac{z^2}{(z-2)^2(z+1)^2} \) are at \( z = \ldots \).

164. Singular points of \( \frac{(z-x)(z+y)}{(z-2)(z-3)} \) are ...... \( \)

165. The poles of \( f(z) = \frac{z^2 + 1}{z^3 + 1} \) are \( z = \ldots \).

166. Residue of \( f(z) = \frac{1 - e^{2z}}{z^3} \) at \( z = 0 \) is ...... \( \)

167. Residue of \( f(z) = z \cos \frac{1}{z} \) at \( z = 0 \) is ...... \( \)

168. Residue of \( f(z) = \frac{\cos z}{z} \) at \( z = 0 \) is ...... \( \)

169. Residue of \( f(z) = \frac{\sin z + z}{\sin z} \) at \( z = 0 \) is ...... \( \)

170. Residue of \( f(z) = e^z \) at \( z = 0 \) is ...... \( \)

171. Residue of \( \tan z \) at \( z = \frac{\pi}{2} \) is ...... \( \)

172. What is the nature of singularity of \( f(z) = \frac{z - \sin z}{z^3} \) at \( z = 0 \)?

173. What is the nature of singularity of \( f(z) = e^{z-a} \) at \( z = a \)?

**PROBABILITY DISTRIBUTIONS**

174. A bag contains 50 tickets numbered 1, 2, 3, 4, ..., 50, of which five are drawn at random and arranged in ascending order of magnitude \( (x_1 < x_2 < x_3 < x_4 < x_5) \). What is the probability that \( x_3 = 30 \)?
A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?

What is the chance that a leap year should have 53 Sundays?

A and B throw alternately with a single die. A having the first throw. The one who first throws ace is to win. What are their respective chances of winning?

Three bags A, B, C contains 4 red, 3 black, 2 white; 3 red, 4 black, 4 white; and 5 red, 2 black, 6 white balls respectively. If a bag is selected at random and a ball is drawn from it, find the probability that the ball drawn is red.

A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. What are the odds against his winning the bet?

A speaks truth in 75% cases and B in 80% cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

The probabilities of A, B, C solving a problem are \( \frac{1}{3}, \frac{2}{7}, \text{ and } \frac{3}{8} \) respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them will solve it.

A family has two children. What is the probability that both the children are boys given that at least one of them is a boy?

A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

If \( P(A) = 0.8, P(B) = 0.5 \) and \( P(\overline{A} \cup \overline{B}) = 0.4 \), find \( P(A \cup B) \).

If \( P(A) = \frac{6}{11}, P(B) = \frac{5}{11} \) and \( P(A \cup B) = \frac{7}{11} \), find \( P(A \cap B) \).

Given that the two numbers appearing on throwing two dice are different, find the probability of the event 'the sum of numbers on the dice is 4.'

If A and B are events such that \( P(\overline{A} \cup B) = P(\overline{A}B) \), then .......

Three cards are drawn successively, without replacement, from a pack of 52 well-shuffled cards. What is the probability that first two cards are kings and the third card is an ace?

Prove that if E and F are independent events, then so are the events \( E' \text{ and } F' \).

A die is thrown thrice. Find the probability of getting an odd number at least once.

Bag I contains 3 red and 4 black balls while another Bag II contains 6 red and 6 black balls. One ball is drawn at random and it is found to be red. Find the probability that it was drawn from Bag II.

Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Write the probability distribution of the number of aces.

The probability distribution of a random variable \( X \) is

\[
\begin{align*}
X & : & 0 & 1 & 2 & 3 & 4 \\
\Pr(X) & : & 0.1 & k & 2k & 2k & k,
\end{align*}
\]

Find \( k \).

Find the variance of the number obtained on a throw of an unbiased die.

Find the mean number of heads in three tosses of a fair coin.

Two dice are thrown simultaneously. If \( X \) denotes the number of sixes, find the expectation of \( X \).
16. \( f(x) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\sin nx}{n} \)  

17. 18. cosine  

19. \( f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \)  

20. \( f(x) = \sin x - \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \)  

21. 0 22. sine 23. cosine 24. even 25. odd 26. even 27. odd 28. See Fourier series, 1.4 30. 0 31. y-axis 32. \( \frac{1}{2} [f(c-0) + f(c+0)] \)  

33. \( \frac{\pi}{2} \) 34. \( \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos \frac{n \pi x}{T} \, dx \) 35. \( \frac{1}{2} a_n \) 36. 0  

37. \( f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( \cos \frac{n \pi x}{2} - \frac{1}{4} \cos \frac{3n \pi x}{2} + \frac{1}{9} \cos \frac{5n \pi x}{2} - \ldots \right) \)  

38. \( f(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n \pi x}{l} \) where \( a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n \pi x}{l} \, dx \) and \( a_0 = \frac{1}{l} \int_{0}^{l} f(x) \, dx \)  

39. \( f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{l} \) where \( b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n \pi x}{l} \, dx \)  

40. \( \frac{4}{\pi} \left( \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right) \)  

41. \( f(x) = \frac{\pi}{2} \int_{0}^{\infty} \sin \lambda x \int_{x}^{\infty} f(t) \sin \lambda t \, dt \, dl \)  

42. \( f(x) = \frac{2}{\pi} \int_{0}^{\infty} \cos \lambda x \int_{x}^{\infty} f(t) \cos \lambda t \, dt \, dl \) 43. \( f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x) \, dt \, dl \)  

44. \( F(s) = \int_{0}^{\infty} f(x) \cos sx \, dx \) 45. \( f(x) = \frac{2}{\pi} \int_{0}^{\infty} F(x) \cos sx \, dx \)  

46. \( \hat{F}(s) = \int_{0}^{\infty} f(x) \sin sx \, dx \) 47. \( f(x) = \frac{2}{\pi} \int_{0}^{\infty} \hat{F}(x) \sin sx \, dx \)  

48. \( F(s) = \int_{0}^{\infty} f(x) e^{sx} \, dx \) 49. \( f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-sx} \, ds \)  

50. \( F_n(n) = \int_{0}^{\infty} f(x) \sin \frac{n \pi x}{c} \, dx \) where \( n \) is an integer.
80. \( e^{x^2 - y^2} \cos(2xy), e^{x^2 - y^2} \sin(2xy) \)
81. \( \frac{1}{2} \log(\tau^2 + \beta^2), 2n \pi + \tan^{-1} \frac{\beta}{\alpha} \)
82. \( \log 3 + i (2a + 1) \pi \)
83. Real part = log 3, Imaginary part = \( 2n \pi + \tan^{-1} \frac{3}{4} \)
84. \( \alpha \)
85. \( \sin x \cos y \)
86. \( \sin \frac{2x}{y} \cos 2x \cos 2y \)
87. \( \frac{2 \cos x \cos y}{\cos 2x \cos 2y} \)
88. \( \sin x \sin y \)
89. \( \frac{\sin 2x}{2x} \cos 2y \cos 2x \cos 2y \)
90. \( \sin x \cosh y \)
91. \( \frac{\sin 2x}{\cos 2x \cosh 2y} \)
92. \( \cosh 2x \cos 2y \)
93. \( \frac{2 \cos x \cos y}{\cos 2x + \cos 2y} \)
94. \( \sinh x \cos y \)
95. \( \frac{x + \sqrt{x^2 + 1}}{1 - x^2 - y^2} \)
96. \( \tan^{-1} \frac{2x}{1 - x^2 - y^2} \)
97. \( \frac{1 + x}{1 - x} \)
98. \( \frac{3}{5} \)
99. \( \frac{3}{5} \)
100. \( (i) \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y} \) or \( \frac{\partial u}{\partial y} - i \frac{\partial v}{\partial x} \)
101. \( (ii) \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \)
102. \( \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \)
103. \( \frac{\partial u}{\partial x} - \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{1}{r} \frac{\partial u}{\partial \theta} \)
104. Any solution of the Laplace’s equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) is called a harmonic function.
105. Laplace’s
106. Harmonic
107. regular, holomorphic
108. constant
109. no point
110. (i)
111. 1
112. \( e^{ix} \sin y + c \)
113. Yes. [Hint. Show that \( (x^2 - y^2) + i (xy) \) is analytic.]
114. \( a = 1, b = -1, c = -1, d = 2 \)
115. \( \alpha \)
116. \( 2x + y + c \)
117. \( 2 \)
118. \( \pm 5 \)
119. \( \frac{2}{3} (-1 + i) \)
120. \( \frac{5}{3} (2 - i) \)
121. 0
122. \( \frac{3}{2} \)
123. \( \frac{1}{2} \)
124. \( \frac{1}{2} \)
125. \( \frac{1}{2} \)
126. \( \frac{1}{2} \)
127. \( \frac{1}{2} \)
128. \( \frac{1}{2} \)
129. \( \frac{1}{2} \)
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161. \( \frac{1}{2} \)
162. \( \frac{1}{2} \)
163. \( \frac{1}{2} \)
164. \( \frac{1}{2} \)
165. \( \frac{1}{2} \)
### APPENDIX—II

**Table 1: NORMAL TABLE**

**AREAS UNDER THE STANDARD NORMAL**

Curve $\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$

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### Additional Table: Mean Values

<table>
<thead>
<tr>
<th>$x$</th>
<th>212.8</th>
<th>$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0040</td>
<td>0.0040</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0080</td>
<td>0.0080</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0120</td>
<td>0.0120</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0240</td>
<td>0.0240</td>
</tr>
<tr>
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<td>0.0280</td>
<td>0.0280</td>
</tr>
<tr>
<td>1.2</td>
<td>0.0320</td>
<td>0.0320</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>3.0</td>
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<td>0.0560</td>
</tr>
<tr>
<td>3.1</td>
<td>0.0600</td>
<td>0.0600</td>
</tr>
</tbody>
</table>

**Mean Values:**

- Mean value of $f(x)$ is approximately 0.0228 for $x = 1.0$.
### Table 2: Significant Values $t_\alpha (\alpha)$ of $t$-Distribution (Two Tail Areas) ($|t| > t_\alpha (\alpha) = \alpha$

<table>
<thead>
<tr>
<th>d.f. (v)</th>
<th>0.50</th>
<th>0.10</th>
<th>0.05</th>
<th>0.02</th>
<th>0.01</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>1.64</td>
<td>2.32</td>
<td>3.18</td>
<td>6.06</td>
<td>63.66</td>
</tr>
<tr>
<td>2</td>
<td>0.82</td>
<td>0.93</td>
<td>1.69</td>
<td>2.24</td>
<td>3.19</td>
<td>12.92</td>
</tr>
<tr>
<td>3</td>
<td>0.77</td>
<td>0.91</td>
<td>1.64</td>
<td>2.44</td>
<td>3.75</td>
<td>7.71</td>
</tr>
<tr>
<td>4</td>
<td>0.74</td>
<td>0.89</td>
<td>1.61</td>
<td>2.58</td>
<td>4.08</td>
<td>5.33</td>
</tr>
<tr>
<td>5</td>
<td>0.73</td>
<td>0.87</td>
<td>1.57</td>
<td>2.68</td>
<td>4.36</td>
<td>4.60</td>
</tr>
<tr>
<td>6</td>
<td>0.72</td>
<td>0.84</td>
<td>1.53</td>
<td>2.78</td>
<td>4.64</td>
<td>4.03</td>
</tr>
<tr>
<td>7</td>
<td>0.71</td>
<td>0.82</td>
<td>1.50</td>
<td>2.85</td>
<td>4.94</td>
<td>3.61</td>
</tr>
<tr>
<td>8</td>
<td>0.71</td>
<td>0.80</td>
<td>1.47</td>
<td>2.94</td>
<td>5.20</td>
<td>3.25</td>
</tr>
<tr>
<td>9</td>
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<td>0.79</td>
<td>1.45</td>
<td>3.03</td>
<td>5.44</td>
<td>2.97</td>
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<tr>
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<td>0.78</td>
<td>1.43</td>
<td>3.12</td>
<td>5.68</td>
<td>2.74</td>
</tr>
<tr>
<td>11</td>
<td>0.70</td>
<td>0.77</td>
<td>1.42</td>
<td>3.22</td>
<td>5.91</td>
<td>2.57</td>
</tr>
<tr>
<td>12</td>
<td>0.70</td>
<td>0.76</td>
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<tr>
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<tr>
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<tr>
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<td>2.08</td>
</tr>
<tr>
<td>17</td>
<td>0.69</td>
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<td>2.02</td>
</tr>
<tr>
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<td>3.88</td>
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</tr>
<tr>
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<tr>
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<td>0.69</td>
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</tr>
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</table>

### Table 3: Chi-Square ($\chi^2$) Significant Values $\chi^2_\alpha (\alpha)$ for Chi-Square Distribution Right Tail Areas

<table>
<thead>
<tr>
<th>Degree of freedom (v)</th>
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<th>0.05</th>
<th>0.02</th>
<th>0.01</th>
</tr>
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<td>.045</td>
<td>.026</td>
<td>.010</td>
</tr>
<tr>
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<td>.0097</td>
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<td>.023</td>
<td>.014</td>
</tr>
<tr>
<td>3</td>
<td>.0115</td>
<td>.0125</td>
<td>.059</td>
<td>.029</td>
<td>.027</td>
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<tr>
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<td>.180</td>
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<td>.162</td>
<td>.187</td>
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<td>.226</td>
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<td>.201</td>
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<td>14.933</td>
<td>11.773</td>
<td>.247</td>
<td>.192</td>
<td>.222</td>
</tr>
</tbody>
</table>

**Note:** For degrees of freedom (v) greater than 30, the quantity $\sqrt{\chi^2 - 2N - 1}$ may be used as a normal variable with unit variance.
APPENDIX—III

SOME IMPORTANT CURVES

Cissoid (Imp.)
y^2(2a - x) = x^3

Catenary
y = a cosh \frac{x}{b}

Cubical Parabola
\( a^2 y = x^3 \)

Semicubical Parabola
\( a y^2 = x^3 \)

Cardioid (Imp.)
\( r = a(1 + \cos \theta) \)

Cycloid (Most Imp.)
\( x = a(\theta - \sin \theta) \)
\( y = a(1 - \cos \theta) \)

Hypocycloid (Imp.)
\( x^3 + y^3 = a^3 \)
\( x = a \cos^3 \theta \)
\( y = a \sin^3 \theta \)

Wilson of Agnesi
\( x^2 + 4y^2 = 2a^2 \)

Systrophoid
\( y^2(a - x) = a^2(a + x) \)

Folium of Descartes
\( x^2 + y^2 = 3axy \)
EXAMINATION PAPERS

M.D.U., Rohtak
(Common to All These Branches)
Examination-May, 2011
Mathematics
Paper: Math-201-F

Time: 3 hours  
Maximum Marks: 100

Before answering the questions, candidates should ensure that they have been supplied the correct and complete question paper. No complaint in this regard, will be entertained after examination.

Note: Attempt five questions in all. Question No. 1 is compulsory. Attempt four questions selecting one question from each Section (A to D). All questions carry equal marks.

1. (a) Express \( f(x) = x \) as a Fourier series in the interval \(-\pi < x < \pi\).

(b) Find the Fourier transforms of \( f(u) = \begin{cases} 1, & |u| < u_0 \\ 0, & |u| > u_0 \end{cases} \)

(c) If the potential function is \( \log(x^2 + y^2) \), find the flux function and the complex potential function.

(d) In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails.

(e) Using graphical method, solve the following L.P.P.

Maximize \( Z = 2x_1 + 3x_2 \)

subject to

\[
\begin{align*}
    x_1 - x_2 & \leq 2 \\
    x_1 + x_2 & \geq 4 \\
    x_1, x_2 & \geq 0
\end{align*}
\]

SECTION-A

2. (a) Find the Fourier expansion for the function

\( f(x) = e^x \) in \(-\pi < x < \pi\).

(b) Obtain the Fourier series expansion for the function \( f(x) = x^2 \) in \([-\pi, \pi]\) and deduce the following from it:

\[
\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots
\]

3. (a) Find the Fourier cosine transform of \( f(x) = \frac{1}{1 + x^2} \).

(b) Verify convolution theorem for \( f(x) = e^{-x^2} \).
SECTION-B

4. (a) If \( \tan(\theta + i\phi) = e^{a} \), show that \( \theta = \left( n + \frac{1}{2} \right) \frac{\pi}{2} \) and
\[
\phi = \frac{1}{2} \log \tan \left( \frac{\pi}{4} \cdot \frac{a}{2} \right)^{1/2}.
\]

(b) If \( \sin^{-1}(z + i) = \log(A + iB) \), show that
\[
\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1,
\]
where \((A^2 + B^2) = e^{2a}

5. (a) Prove that the function \( f(z) \) defined by:
\[
f(z) = \frac{x^2(1+i) - y^2(1-i)}{x^2 + y^2} (x \neq 0),
\]

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet \( f'(0) \)
does not exist.

(b) Evaluate \( \int \frac{e^z}{(e^z + z^2)} \frac{dz}{C} \) where \( C \) is \( |z| = 4 \).

SECTION-C

6. (a) Find Taylor's expansion of \( f(z) = \frac{2x^2 + 1}{x^2 + z} \) about the point \( z = i \).

(b) What type of singularity have the following functions:

(i) \( e^{z^2} \),

(ii) \( z^{-3+i} \),

(iii) \( \frac{1}{1-e^z} \)

7. (a) In a lottery, \( m \) tickets are drawn at a time out of \( n \) tickets numbered from 1 to \( n \). Find
the expected value of the sum of the numbers on the tickets drawn.

(b) Fit a normal curve to the following distribution:

<table>
<thead>
<tr>
<th>( x )</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

SECTION-D

8. (a) A group of 10 rats fed on a diet \( A \) and another group of 8 rats fed on a different diet \( B \),
recorded the following increase in weights:

<table>
<thead>
<tr>
<th>Diet A:</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>3</th>
<th>9</th>
<th>6</th>
<th>10 gm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diet B:</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>1</td>
<td>2</td>
<td>8 gm</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Does it show the superiority of diet \( A \) over that of \( B \)?

(b) The means of simple samples of sizes 1.000 and 2.000 are 675 and 680 cm respectively.
Can the samples be regarded as drawn from the same population of S.D. 2.5 m?

9. (a) Using dual simplex method, solve the following problem:

Minimize \( Z = 2x_1 + 2x_2 + 4x_3 \), subject to

\[
2x_1 + 3x_2 + 5x_3 \geq 2, 3x_1 + x_2 + 7x_3 \leq 3 \]
\[
x_1 + 4x_2 + 6x_3 \leq 5, x_1, x_2, x_3 \geq 0.
\]

(b) Solve the following L.P.P. by simplex method:

Minimize \( Z = x_1 - 3x_2 + 3x_3 \), subject to

\[
3x_1 - x_2 + 2x_3 \leq 7, 2x_1 + 4x_2 - 12 = 4x_1 + 3x_2 + 8x_3 \leq 10, x_1, x_2, x_3 \geq 0.
\]
2. (a) Show that for \(-\pi \leq x \leq \pi\),
\[
\cos cx = \frac{\sin cx}{\pi} \left[ 1 - \frac{2c \cos x}{c^2 - 1} + \frac{2c \cos 2x}{c^2 - 2^2} + \ldots \right]
\]
where \(c\) is non-integral, hence deduce that
\[
\pi \cosec cx = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{n + c} - \frac{1}{n + 1 - c} \right]
\]
(b) Find the Fourier series to represent \(f(x) = x^2 - 2\) when \(-2 \leq x \leq 2\).
(c) Find the Fourier sine and cosine transform of the function \(e^{-x^2}\).
(d) State and prove convolution theorem for Fourier transforms.

UNIT-B

4. (a) If \(C \tan (x + iy) = A + iB\) prove that
\[
\tan 2x = \frac{2CA}{C^2 - A^2 - B^2}
\]
(b) Determine the analytic function whose real part is \(e^{iz} (\cos 2y - \sin 2y)\).

5. (a) If \(f(z)\) in a regular function of \(z\), prove that
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2.
\]
(b) If \(f(z) = \int_{z}^{z^2 + 72 + 1} \frac{3x^2 + 72 + 1}{z - \frac{1}{2}} dx\), where \(c\) is the circle \(x^2 + y^2 = 4\). Find the value of \(f''(1 - i)\).

UNIT-C

6. (a) Show that when \(|z + 1| < 1\),
\[
z^2 = 1 + \sum_{n=1}^{\infty} (n + 1)(z + 1)^n
\]
(b) Evaluate
\[
\int_{0}^{2\pi} \frac{d\theta}{1 - 2a \sin \theta + a^2}, 0 < a < 1.
\]

7. (a) The probability that a man aged 50 years will die within a year is 0.0125. What is the probability that out of 12 such men, at least 11 will reach their fifty first birthday?
(b) Prove that mean deviation from the mean of a normal distribution is \(\frac{4}{5}\) of its standard deviation.

### Mathematics - III

**M.D.U., Rohtak**

**B.Tech. 3rd Semester (Electrical Engg.)**

**Branch-1 Examination-December, 2011**

**Time allowed: 3 hours | Maximum Marks: 100**

**Note:** Attempt five questions in total selecting one question from each Unit. Q. No. 1 is compulsory. All questions carry equal marks.

1. (a) Find the Fourier series of the function defined by:
\[
f(x) = \begin{cases} 
0, & -\pi \leq x < 0 \\
\pi, & \pi \leq x \leq \pi 
\end{cases}
\]
(b) Solve the integral equation:
\[
\int_{-\pi}^{\pi} f(x) \cos \lambda x \, dx = e^{-\lambda}
\]
(c) Find regular function whose imaginary part is \(\frac{x-y}{x^2+y^2}\).
(d) Evaluate \(\int_{c} (z - z^3) \, dz\), where \(c\) is upper half of the circle.
(e) Evaluate \(\int_{c} \frac{(z^2 - 2z + 1)}{z - 1} \, dz\), where \(c\) is the circle \(|z| = \frac{1}{2}\).
(f) If a random variable has a Poisson distribution such that \(P(1) = P(2)\), find mean of the distribution.
(g) The average marks in English of a sample of 100 is 51 with a SD of 6 marks. Could this have a random sample from a population with average marks 50? (h) Intelligence tests given of two groups of boys and girls:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>S.D</th>
<th>Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Girls</td>
<td>75</td>
<td>8</td>
<td>60</td>
</tr>
<tr>
<td>Boys</td>
<td>73</td>
<td>10</td>
<td>100</td>
</tr>
</tbody>
</table>

Examine if the difference between mean scores is significant.
UNIT-D

8. (a) Using simplex method

Maximize \[ z = x_1 + 2x_2, \]
Subject to \[ 2x_1 + x_2 \leq 8, \]
\[ 2x_1 + 3x_2 \leq 12, \]
\[ x_1, x_2 \geq 0 \]

(b) Obtain the dual of:

Maximize \[ z = 5x_1 + 3x_2, \]
Subject to \[ x_1 + x_2 \leq 2, \]
\[ 5x_1 + 2x_2 \leq 10, \]
\[ 3x_1 + 8x_2 \leq 12, \]
\[ x_1, x_2 \geq 0 \]

9. (a) The 9 items of a sample have the following values 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these values differ significantly from the assumed mean 47.5? (b) A certain stimulus administered to each of 12 patients resulted in the following increases of blood pressures: 5, 2, 8, -1, 3, 0, -2, 1, 5, 0, 4, 6. Can it be concluded that the stimulus will in general be accompanied by an increase of blood pressure?

K.U., Kurukshetra
B.Tech. Semester III (BT-3/DX)
Mathematics-III
Paper: Math-201 (E)

Time: Three hours

[Maximum Marks: 100]

Notes: Attempt five questions in all, selecting at least one question from each unit. All questions carry equal marks.

UNIT-I

1. (a) Obtain a Fourier series for the function \( f(x) = x \sin x, 0 < x < 2\pi \).

(b) Express \( f(x) = x \) as a half range cosine series in \( 0 < x < 2 \).

2. (a) Find the Fourier transform of \( e^{-x^2/2}, -\infty < x < \infty \).

(b) State and prove Convolution theorem for Fourier transforms.

UNIT-II

3. (a) If \( \tan (\theta + i\phi) = \tan \alpha + i \sec \alpha \), prove that \[ e^{2\theta} = \pm \cot \frac{\alpha}{2} \] and \( 2\theta = \left( n + \frac{1}{2} \right) \pi + \alpha \).

(b) Define an analytic function state and prove Cauchy-Riemann conditions for an analytic function.

4. (a) If \( f(x) = u + iv \) is an analytic function, find \( f(x) \), if \( u - v = e^x (\cos y - \sin y) \).

(b) If \( \omega = \phi + iv \) represents the complex potential for an electric field and \( \psi = x^2 - y^2 + \frac{x}{x^2 + y^2} \), determine the function \( \phi \).

UNIT-III

5. (a) In a bolt factory, machines A, B and C manufacture 25%, 35% and 40% of the total output. 5%, 4% and 2% of the total output are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B or C?

(b) Find the moment generating function of the exponential distribution \( f(x) = \frac{1}{c} e^{-x/c} \), \( 0 \leq x \leq \infty, c > 0 \). Hence find its mean and S.D.
6. (a) In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least three defective parts.
(b) In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the Mean and S.D. of the distribution.

UNIT-IV

7. (a) Solve the following LPP Graphically:
Maximize \[ Z = 5x_1 + 3x_2 \]
subject to \[ 4x_1 + 5x_2 \leq 1000, \]
\[ 3x_1 + 8x_2 \leq 1200, \]
\[ 5x_1 + 2x_2 \leq 1000; \]
\[ x_1, x_2 \geq 0. \]

(b) A firm manufactures two items A and B. It purchases castings which are then machined, bored and polished. Castings for items A and B cost \( \text{₹} \) 3 and \( \text{₹} \) 4 each and are sold at \( \text{₹} \) 6 and \( \text{₹} \) 7 respectively. Running costs of these machines are \( \text{₹} \) 20, \( \text{₹} \) 14 and \( \text{₹} \) 17.50 per hour respectively. Formulate the problem so that the product mix maximizes the profit. The capacities of the machines are

<table>
<thead>
<tr>
<th>Item</th>
<th>Machining</th>
<th>Boring</th>
<th>Polishing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Item-A</td>
<td>25 per hr.</td>
<td>28 per hr.</td>
<td>35 per hr.</td>
</tr>
<tr>
<td>Item-B</td>
<td>40 per hr.</td>
<td>35 per hr.</td>
<td>25 per hr.</td>
</tr>
</tbody>
</table>

8. (a) Using Simplex method
Maximize \[ Z = 5x_1 + 3x_2 \]
subject to \[ x_1 + x_2 \leq 2, \]
\[ 5x_1 + 2x_2 \leq 10, \]
\[ 3x_1 + 8x_2 \leq 12; \]
\[ x_1, x_2 \geq 0. \]

(b) Using dual Simplex method
Maximize \[ Z = -3x_1 - x_2 \]
subject to \[ x_1 + x_2 \geq 1, \]
\[ 2x_1 + 3x_2 \geq 2, \]
\[ x_1, x_2 \geq 0. \]
ABOUT THE BOOK
The author has endeavoured to present the fundamental concepts of mathematics in a comprehensive and lucid manner. An outstanding and distinguishing feature of the book is the large number of typical solved examples followed by well graded exercises for practice. Many examples and problems have been selected from recent papers (2005 onwards) of various engineering examinations.
The book is strictly in accordance with the latest syllabi.

ABOUT THE AUTHOR
N.P. Bali is a prolific author of over 100 books for degree and engineering students. He has been writing books for more than forty years.
His books on the following topics are well known for their easy comprehension and lucid presentation: Algebra, Trigonometry, Differential Calculus, Integral Calculus, Real Analysis, Co-ordinate Geometry, Statics, Dynamics etc.a